CHAPTER 5

Continuous Functions

5.1. Continuous Functions

Definition (5.1.1). Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, and $c \in A$. $f$ is continuous at $c$ if

\[
\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in A \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.
\]

Theorem (5.1.2). $f : A \rightarrow \mathbb{R}$ is continuous at $c \in A$ if and only if for each neighborhood $V_{\epsilon}(f(c))$ of $f(c)$ there exists a neighborhood $V_{\delta}(c)$ of $c$ such that

\[
x \in A \cap V_{\delta}(c) \implies f(x) \in V_{\epsilon}(f(c)), \text{ i.e.,}
\]

\[
f(A \cap V_{\delta}(c)) \subseteq V_{\epsilon}(f(c)).
\]

Note.

1. $\delta = \delta(\epsilon, c)$ usually.

2. If $c$ is a cluster point of $A$,

\[
f \text{ is continuous at } c \iff \lim_{x \to c} f(x) = f(c).
\]

3. If $c$ is not a cluster point of $A$, i.e., an isolated point of $A$, then $f$ is continuous at $c$ since $\exists \delta > 0 \implies A \cap V_{\delta}(c) = \{c\}$.

Theorem (5.1.3 — Sequential Criterion). $f : A \rightarrow \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence $(x_n)$ in $A$ converging to $c$, the sequence $(f(x_n))$ converges to $f(c)$. 

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THEOREM (5.1.4 — Discontinuity Criterion). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A$. $f$ is discontinuous at $c$ if

$\exists$ a sequence $(x_n)$ in $A \ni \lim(x_n) = c$ but $\lim (f(x_n)) \neq f(c)$.

DEFINITION (5.1.5). Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. If $B \subseteq A$, $f$ is continuous on $B$ if $f$ is continuous at every point of $B$.

EXAMPLE.

(1) $f(x) = L$, $L \in \mathbb{R}$, $x \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = \lim_{x \to c} L = L = f(c) \forall c \in \mathbb{R},$$

so $f$ is continuous on $\mathbb{R}$.

(2) $f(x) = x$

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c = f(c),$$

so $f$ is continuous on $\mathbb{R}$.

(3) It can be shown that

$x - \frac{1}{6}x^3 \leq \sin x \leq x$ for $x \geq 0$ and $x \leq \sin x \leq x - \frac{1}{6}x^3$ for $x \leq 0$.

Thus

$$1 - \frac{1}{6}x^2 \leq \frac{\sin x}{x} \leq 1 \forall x \in \mathbb{R}.$$ 

Since

$$\lim_{x \to 0} \left(1 - \frac{1}{6}x^2\right) = \lim_{x \to 0} 1 = 1,$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$
5.1. Continuous Functions

**Definition.** If \( f \) is discontinuous at \( c \), but \( \lim_{x \to c} f(x) \) exists, then \( x = c \) is called a **removable discontinuity** of \( f \) since

\[
g(x) = \begin{cases} 
  f(x), & \text{if } x \neq c \\
  \lim_{x \to c} f(x), & \text{if } x = c
\end{cases}
\]

is continuous at \( c \).

**Note.** We are defining or redefining \( f(c) \).

Thus

\[
g(x) = \begin{cases} 
  \sin \frac{x}{x}, & \text{if } x \neq 0 \\
  1, & \text{if } x = 0
\end{cases}
\]

is continuous at \( x = 0 \) (on \( \mathbb{R} \), in fact).

(4) The **characteristic function** of the set \( A \subseteq \mathbb{R} \) is

\[
\chi_A(x) = \begin{cases} 
  1, & \text{if } x \in A \\
  0, & \text{if } x \not\in A.
\end{cases}
\]

\[
\chi(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 
  1, & \text{if } x \in \mathbb{Q} \\
  0, & \text{if } x \not\in \mathbb{Q}
\end{cases}
\]

is discontinuous everywhere since every neighborhood of \( c \in \mathbb{R} \) contains infinitely many rationals and irrationals.

(5) A function continuous at \( x = 0 \) only is

\[
f(x) = x \chi(x).
\]

This has already been proved as Page 110 #15.

Also, for \( \{a_1, a_2, \ldots, a_n\} \), \( a_i \neq a_j \) for \( i \neq j \),

\[
f(x) = \chi(x)(x - a_1)(x - a_2) \cdots (x - a_n)
\]

is continuous on \( \{a_1, a_2, \ldots, a_n\} \), but nowhere else.
(6) The Ruler (Thomae’s) function

\[ f(x) = \begin{cases} 
0, & \text{if } x \not\in \mathbb{Q} \\
\frac{1}{n}, & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{m}{n}, n \in \mathbb{N}, \frac{m}{n} \text{ in lowest terms}
\end{cases} \]

is continuous at each irrational and discontinuous at each rational (graph is on Page 127).

**Proof.**

(a) Suppose \( c \in \mathbb{Q} \). Let \( (x_n) \) be a sequence of irrationals with \( \lim(x_n) = c \). Then \( \lim (f(x_n)) = 0 \neq f(c) \). thus \( f(x) \) is discontinuous at \( x = c \).

(b) Suppose \( c \in \mathbb{R} \setminus \mathbb{Q} \) and \( \epsilon > 0 \). By the Archimedean Property,

\[ \exists n_0 \in \mathbb{N} \ni \frac{1}{n_0} < \epsilon \]. Now \( \exists \) only a finite number of rationals in \((c - 1, c + 1)\) with denominator less than \( n_0 \).

Choose \( \delta > 0 \ni (c - \delta, c + \delta) \) contains no rationals with denominator less than \( n_0 \). then

\[ x \in \mathbb{R} \text{ and } |x - c| < \delta \implies |f(x) - f(c)| = |f(x)| = f(x) \leq \frac{1}{n_0} < \epsilon. \]

Thus \( f \) is continuous at \( x = c \). \( \square \)

(7) \( f(x) = x \sin \left( \frac{1}{x} \right) \). Is the discontinuity at \( x = 0 \) removable?

Yes, since \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0 \).

**Proof.**

\[ |f(x) - 0| = \left| x \sin \left( \frac{1}{x} \right) \right| \leq |x| \to 0 \text{ as } x \to 0. \]

Then

\[ f(x) = \begin{cases} 
x \sin \left( \frac{1}{x} \right), & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases} \]

is continuous on \( \mathbb{R} \). \( \square \)
(8) Any polynomial $p(x)$ is continuous on $\mathbb{R}$ since $\lim_{x \to c} p(x) = p(c)$.

(9) Any rational function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, is continuous on $\{x : q(x) \neq 0\}$ since $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$ on this set.

(10) $f(x) = \frac{x^2 - 1}{x - 1}$ is continuous except at $x = 1$. Since
\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2,
\]
the discontinuity at $x = 1$ is removable. Thus
\[
F(x) = \begin{cases} 
  \frac{x^2 - 1}{x - 1}, & \text{if } x \neq 1 \\
  2, & \text{if } x = 1
\end{cases} = x + 1
\]
is continuous on $\mathbb{R}$.

**Homework**

Pages 129-30 # 5, 7, 8, 12

Pag 134 # 8 (Hint: Use Problem 8 from Page 129)
5.2. Combinations of Continuous Functions

**Theorem (5.2.1).** Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and $b \in \mathbb{R}$. Suppose $c \in A$ and $f$ and $g$ are continuous at $c$.

(a) Then $f \pm g$, $fg$, and $bf$ are continuous at $c$.

(b) If $g(x) \neq 0 \ \forall \ x \in A$, then $\frac{f}{g}$ is continuous at $c$.

**Example.**

(1) $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$.

**Proof.** Facts from Trigonometry:

$$|\sin z| \leq |z|, \quad |\cos z| \leq 1, \quad |\sin z| \leq 1$$

$$\sin x - \sin y = 2 \sin \left[ \frac{1}{2} (x - y) \right] \cos \left[ \frac{1}{2} (x + y) \right]$$

$$\cos x - \cos y = 2 \sin \left[ \frac{1}{2} (x + y) \right] \sin \left[ \frac{1}{2} (y - x) \right]$$

Then:

$$|\sin x - \sin c| = 2 \cdot |\sin \frac{1}{2} (x - c)| \cdot |\cos \frac{1}{2} (x + c)| \leq 2 \cdot \frac{1}{2} |x - c| \cdot 1 = |x - c|$$

and

$$|\cos x - \cos c| = 2 \cdot |\sin \frac{1}{2} (x + c)| \cdot |\sin \frac{1}{2} (c - x)| \leq 2 \cdot 1 \cdot \frac{1}{2} |c - x| = |x - c|.$$
5.2. COMBINATIONS OF CONTINUOUS FUNCTIONS

**Theorem (5.2.5).** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $f(x) \geq 0$ $\forall$ $x \in A$. Define $\sqrt{f} : A \to \mathbb{R}$ by $\sqrt{f}(x) = \sqrt{f(x)}$. If $f$ is continuous at $c \in A$, then $\sqrt{f}$ is continuous at $c$.

Recall $(g \circ f)(x) = g[f(x)]$. This requires range $f \subseteq$ domain $g$.

**Theorem (5.2.6).** Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and $f(A) \subseteq B$. If $f$ is continuous at $c \in A$ and $g$ is continuous at $b = f(c) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at $c$.

**Proof.** Let $\epsilon > 0$ be given. Since $g$ is continuous at $f(c)$,

$$\exists \gamma > 0 \Rightarrow y \in B \text{ and } |y - f(c)| < \gamma \implies \left| g(y) - g[f(c)] \right| < \epsilon.$$ 

Since $f$ is continuous at $c$,

$$\exists \delta > 0 \Rightarrow x \in A \text{ and } |x - c| < \delta \implies \left| f(x) - f(c) \right| < \gamma \implies \left| g[f(x)] - g[f(c)] \right| < \epsilon.$$ 

Thus $g \circ f$ is continuous at $c$. \hfill $\square$

**Homework**

Pages 133-4 ∆ 1 (just give the points of continuity), 3, 7
5.3. Continuous Functions on Intervals

**Definition (5.3.1).** \( f : A \to \mathbb{R} \) is bounded on \( A \) if
\[
\exists M > 0 \iff |f(x)| \leq M \forall x \in A.
\]

**Note.** Continuous functions are not necessarily bounded.

**Theorem (5.3.2 — Boundedness Theorem).** Let \( I = [a, b] \) be a closed, bounded interval and let \( f : I \to \mathbb{R} \) be continuous on \( I \). Then \( f \) is bounded on \( I \).

**Proof.** Suppose \( f \) is not bounded on \( I \). Then, \( \forall n \in \mathbb{N}, \exists x_n \in I \iff |f(x_n)| > n \). Since \( I \) is bounded, \( X = (x_n) \) is bounded. By the B-W Theorem, \( X \) has a subsequence \( X' = (x_{n_k}) \iff \lim(x_{n_k}) = x \in I \) (by Theorem 3.2.6). Since \( f \) is continuous at \( x \),
\[
\lim (f(x_{n_k})) = f(x). \quad \text{But then}
\]
\[
(f(x_{n_k})) \text{ is bounded, contradicting that}
\]
\[
|f(x_{n_k})| > n_k \geq k \forall k \in \mathbb{N}.
\]
Thus \( f \) must be bounded on \( I \). \qed

**Definition (5.3.3).** Let \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \). \( f \) has an absolute maximum on \( A \) if
\[
\exists x^* \in A \iff f(x^*) \geq f(x) \forall x \in A.
\]
\( f \) has an absolute minimum on \( A \) if
\[
\exists x_* \in A \iff f(x_*) \leq f(x) \forall x \in A.
\]
Theorem 5.3.4 — Maximum-Minimum Theorem. Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ has an absolute maximum and an absolute minimum on $I$.

Proof. $f(I)$ is nonempty and bounded (by the previous theorem). Then $s^* = \sup f(I)$ and $s_* = \inf f(I)$ exist.

[To show $\exists x^*, x_* \in I \ni f(x^*) = s^*$ and $f(x_*) = s_*$. Then $f(x^*)$ is the absolute maximum and $f(x_*)$ is the absolute minimum on $I$.]

(a) Since $s^* = \sup f(I)$, $\forall n \in \mathbb{N}$, $\exists x_n \in I \ni$

\[ (#) \quad s^* - \frac{1}{n} < f(x_n) \leq s^* . \]

Since $I$ is bounded, $(x_n)$ is bounded, so has a subsequence $(x_{n_k}) \ni \lim (x_{n_k}) = x^* \in I$ (from B-W and Theorem 3.2.6). But $f$ is continuous at $x^*$, so

\[ \lim (f(x_{n_k})) = f(x^*) . \]

By (#),

\[ s^* - \frac{1}{n_k} < f(x_{n_k}) \leq s^* \quad \forall k \in \mathbb{N} . \]

By the Squeeze Theorem,

\[ f(x^*) = \lim (f(x_{n_k})) = s^* . \]

(b) Similar. \qed
**Theorem (5.3.7 — Bolzano’s Intermediate Value Theorem).** Let $I$ be an interval and $f : I \to \mathbb{R}$ be continuous on $I$. If $a, b \in I$ and $k \in \mathbb{R}$ $f(a) < k < f(b)$, then $\exists c \in I$ between $a$ and $b$ $f(c) = k$.

**Proof.** WLOG, assume $a < b$ (the other case is similar).

Let $S = \{x \in [a, b] : f(x) < k\}$.

Since $a \in S$, $S \neq \emptyset$. By the Supremum Theorem,

$\exists c \in [a, b] \Rightarrow c = \sup S$. Then

$$\forall n \in \mathbb{N}, \exists x_n \in S \Rightarrow c - \frac{1}{n} < x_n \leq c.$$ 

Thus $\lim(x_n) = c$ by the Squeeze Theorem, and since $f(x_n) < k \ \forall \ n \in \mathbb{N}$, $\lim(f(x_n)) \leq k$. Since $f$ is continuous at $c$,

$$f(c) = \lim(f(x_n)) \leq k.$$ 

[To show $f(c) \geq k$ also $\Rightarrow f(c) = k$.]

Now let $y_n = \min \left\{ b, c + \frac{1}{n} \right\} \ \forall \ n \in \mathbb{N}$.

Since $c < y_n \leq c + \frac{1}{n}$, $\lim(y_n) = c$ by the Squeeze Theorem. [Where is $y_n$?]

Now, $\forall \ n \in \mathbb{N}$, $y_n \in [a, b]\backslash S$, so $f(y_n) \geq k$.

Again, by continuity at $c$,

$$f(c) = \lim(f(y_n)) \geq k \Rightarrow f(c) = k.$$
Corollary (5.3.8). Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. If

$$\inf f(I) \leq k \leq \sup f(I),$$

then $\exists \ c \in I \ni f(c) = k$.

Proof. From the Maximum-Minimum Theorem, $\exists \ c^*, \ c^* \in I \ni$

$$\inf f(I) = f(c^*) \leq k \leq f(c^*) = \sup f(I)$$

The conclusion follows from Bolzano’s IVT. □

Theorem (3.5.9). Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then the set $f(I) = \{f(x) : x \in I\}$ is a closed, bounded interval.

Proof. $m = \inf f(I) \in f(I)$ and $M = \sup f(I) \in f(I)$ by the Maximum-Minimum Theorem. Clearly, $f(I) \subseteq [m, M]$. If $k \in [m, M]$, by Corollary 5.3.7, $\exists \ c \in I \ni k = f(c)$. Then $k \in f(I)$, so $[m, M] \subseteq f(I)$. Thus $f(I) = [m, M]$. □

Theorem (5.3.10 — Preservation of Intervals Theorem). If $I$ is an interval and $f : I \to \mathbb{R}$ is continuous on $I$, then $f(I)$ is an interval.

Proof. Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$. Then $\exists \ a, b \in I \ni \alpha = f(a)$ and $\beta = f(b)$. From Bolzano’s IVT,

$$k \in (\alpha, \beta) \implies \exists \ c \in I \ni k = f(c) \in f(I).$$

Thus $[\alpha, \beta] \subseteq f(I)$, so $f(I)$ is an interval by Theorem 2.5.1. □

Homework

Pages 140-41 # 1, 3, 5
5.4. Uniform Continuity

Example. 

(1) $f(x) = \frac{1}{x^2}$ is continuous on $(0, \infty)$. Let $u > 0$ and $\varepsilon > 0$ be given. 
Since $f$ is continuous at $u$, $\exists \delta > 0 \ni |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$. 

[How to pick $\delta$?]

$$|f(x) - f(u)| = \left| \frac{1}{x^2} - \frac{1}{u^2} \right| = \frac{|u^2 - x^2|}{x^2 u^2} = \frac{|x + u|}{x^2 u^2} |x - u|.$$ 

Take $\delta \leq \frac{u}{2}$. Then 

$$|x - u| < \delta \implies |x - u| < \frac{u}{2} \implies \frac{u}{2} < x - u < \frac{u}{2} \implies \frac{u}{2} < x < \frac{3u}{2} \implies \frac{3u}{2} < x + u < \frac{5u}{2}.$$ 

Then 

$$|f(x) - f(u)| = \frac{|x + u|}{x^2 u^2} |x - u| < \frac{(\frac{5u}{2})}{(\frac{u}{2})^2 u^2} |x - u| = \frac{10}{u^3} |x - u| < \varepsilon$$ 

if $|x - u| < \frac{\varepsilon u^3}{10}$ also. So take $\delta = \min \left\{ \frac{u}{2}, \frac{\varepsilon u^3}{10} \right\}$. Notice $\delta = \delta(u, \varepsilon)$. 

Suppose $\varepsilon = \frac{1}{4}$. 

If $u = 3$, take $\delta = \min \left\{ \frac{3}{2}, \frac{9}{40} \right\} = .225$. 

If $u = \frac{1}{2}$, take $\delta = \min \{.25, .000390625\} = .000390625$. 

If $u = \frac{1}{100}$, take $\delta = \min \{.005, .000000025\} = .000000025$. 

It turns out to be very useful to know when $\delta = \delta(\varepsilon)$ only, without dependence on $u$. 

**Definition (5.4.1).** Let \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \). \( f \) is uniformly continuous (u.c.) on \( A \) if 
\[
\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0 \ \text{s.t.} \ \forall x, u \in A \text{ and } |x - u| < \delta(\epsilon) \implies |f(x) - f(u)| < \epsilon.
\]

**Note.** Uniform continuity \( \implies \) continuity.

**Example.**

1. \( f(x) = \frac{1}{x^2} \) is u.c. on \([a, \infty)\) where \( a > 0 \).

**Proof.** Let \( \epsilon > 0 \) be given. Then 
\[
|f(x) - f(u)| = \frac{|x + u|}{x^2 u^2} |x - u| = \frac{x + u}{x^2 u^2} |x - u| = \\
\left( \frac{1}{x u^2} + \frac{1}{x^2 u} \right) |x - u| \leq \left( \frac{1}{a^3} + \frac{1}{a^3} \right) |x - u| = \frac{2}{a^3} |x - u| < \epsilon
\]
if \( |x - u| < \frac{a^3 \epsilon}{2} \). Take \( \delta = \frac{a^3 \epsilon}{2} \). Then 
\[
x, u \in A \text{ and } |x - u| < \delta \implies |f(x) - f(u)| < \epsilon.
\]
So \( f \) is u.c. on \([a, \infty)\). \( \square \)

**Theorem (5.4.2 — Nonuniform Continuity Criterion).** Let \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \). TFAE:

1. \( f \) is not u.c. on \( A \).
2. \( \exists \epsilon_0 > 0 \ \forall \delta > 0 \ \exists x_\delta, u_\delta \in A \ \text{s.t.} \ |x_\delta - u_\delta| < \delta \text{ and } |f(x_\delta) - f(u_\delta)| \geq \epsilon_0. \)
3. \( \exists \epsilon_0 > 0 \text{ and two sequences } (x_n) \text{ and } (u_n) \text{ in } A \ \text{s.t.} \ \lim(x_n - u_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \epsilon_0 \ \forall n \in \mathbb{N}. \)
EXAMPLE.

(2) $f(x) = \frac{1}{x^2}$ is not u.c. on $(0, \infty)$ (or on $(0, a]$).

**Proof.** Let $x_n = \frac{1}{\sqrt{n}}$ and $u_n = \frac{1}{\sqrt{n + 1}}$.

$$|x_n - u_n| = \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n + 1}} \right| \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n + 1}} \to 0 \text{ as } n \to \infty,$$

but

$$|f(x_n) - f(u_n)| = |n - (n + 1)| = 1 \forall n \in \mathbb{N}. \tag*{\square}$$

**Theorem (5.4.3 — Uniform continuity Theorem).** Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ is uniformly continuous on $I$.

**Proof.** Suppose $f$ is not u.c. on $I$. Then $\exists \epsilon_0 > 0$ and two sequences $(x_n)$ and $(u_n)$ in $A \ni (x_n - u_n) < \frac{1}{n}$ and $|f(x_n) - f(u_n)| \geq \epsilon_0 \forall n \in \mathbb{N}$.

Since $I$ is bounded, $(x_n)$ is bounded, so by B-W and Theorem 3.2.6

$\exists$ a subsequence $(x_{n_k})$ of $(x_n) \ni \lim(x_{n_k}) = z \in I$. Since

$$|u_{n_k} - z| = |u_{n_k} - x_{n_k} + x_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z| \to 0,$$

$\lim(u_{n_k}) = z$ also. Since $f$ is continuous at $z$,

$$\lim(f(x_{n_k})) = f(z) = \lim(f(u_{n_k})),
$$

so

$$|f(x_{n_k}) - f(u_{n_k})| = |f(x_{n_k}) - f(z) + f(z) - f(u_{n_k})| \leq$$

$$|f(x_{n_k}) - f(z)| + |f(z) - f(u_{n_k})| \to 0,$$

contradicting $|f(x_{n_k}) - f(u_{n_k})| \geq \epsilon_0 \forall k \in \mathbb{N}$.

Thus $f$ is u.c. on $I$. \tag*{\square}
Example.

(3) \( f(x) = x^{73} \) is u.c. on \([-13, 13]\).

(4) \( f(x) = \sqrt{x} \) is u.c. on \([0, 400]\).

(5) \( f(x) = x^{17} \sin e^x - e^{4x} \cos(2x) \) is u.c. on \([-8\pi, 8\pi]\).

(6) \( f(x) = \frac{1}{x^6} \) is u.c. on \( \left[ \frac{1}{4}, 44 \right] \).

What if \( f \) is continuous on a set that is not a closed, bounded interval?

Definition (5.4.4). Let \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \). If

\[
\exists K > 0 \Rightarrow |f(x) - f(u)| \leq K|x - u| \quad \forall x, u \in A,
\]

then \( f \) is a Lipschitz function (or satisfies a Lipschitz condition) on \( A \).

Note. This is equivalent to

\[
\left| \frac{f(x) - f(u)}{x - u} \right| \leq K
\]
on \( A \), i.e., the slopes with all line segments with endpoints in \( A \) is bounded by \( K \).

Theorem (5.4.5). If \( f : A \to \mathbb{R} \) is a Lipschitz function on \( A \), then \( f \) is uniformly continuous on \( A \).

Proof. Given \( \epsilon > 0 \). Suppose \( f \) is Lipschitz with constant \( K \).

Take \( \delta = \frac{\epsilon}{K} \). Then

\[
x, u \in A \text{ and } |x - u| < \delta \implies |f(x) - f(u)| \leq K|x - u| < K \cdot \frac{\epsilon}{K} = \epsilon.
\]

Thus \( f \) is u.c. on \( A \). \( \square \)
Example.

(7) \( f(x) = x^{1/3} \) is u.c. on \((1, \infty)\).

Proof.

\[
|f(x) - f(u)| = |x^{1/3} - u^{1/3}| = \frac{|x - u|}{x^{2/3} + x^{1/3}u^{1/3} + u^{2/3}} < \frac{1}{3}|x - u|.
\]

Thus \( f \) is Lipschitz and u.c. on \((1, \infty)\).

(8) \( g(x) = \sqrt{x} \) is u.c. on \([0, 2]\) but not Lipschitz since

\[
|\sqrt{x} - \sqrt{u}| = \frac{1}{\sqrt{x} + \sqrt{u}}|x - u|
\]

and \( \frac{1}{\sqrt{x} + \sqrt{u}} \) is unbounded as \( x, u \to 0 \).

Thus u.c. \( \not\Rightarrow \) Lipschitz.

Problem (Page 148 # 12). If \( f \) is continuous on \([0, \infty)\) and u.c. on \([a, \infty)\) for some \( a > 0 \), then \( f \) is u.c. on \([0, \infty)\).

Proof. Let \( \epsilon > 0 \) be given. Since \( f \) is continuous on \([0, \infty)\),

\( f \) is u.c. on \([0, a + 1]\). Thus

\[ 0 < \delta_1 < 1 \implies x, u \in [0, a + 1] \text{ and } |x - u| < \delta_1 \implies |f(x) - f(u)| < \epsilon. \]

Since \( f \) is u.c. on \([a, \infty)\),

\[ 0 < \delta_2 < 1 \implies x, u \in [a, \infty) \text{ and } |x - u| < \delta_2 \implies |f(x) - f(u)| < \epsilon. \]

Let \( \delta = \min\{\delta_1, \delta_2\} \). Then

\[ |x - u| < \delta \implies x, u \in [0, a + 1] \text{ or } x, u \in [a, \infty) \implies |f(x) - f(u)| < \epsilon. \]

Thus \( f \) is u.c. on \([0, \infty)\).
What about u.c. on open intervals?

**Theorem (5.4.7).** If $f : A \to \mathbb{R}$ is u.c. and if $(x_n)$ is a Cauchy sequence in $A$, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

**Proof.** Let $(x_n)$ be a Cauchy sequence in $A$ and $\epsilon > 0$ be given. Since $f$ is u.c. on $A$,

$$\exists \delta > 0 \ni x, u \in A \text{ and } |x - u| < \delta \implies |f(x) - f(u)| < \epsilon.$$ 

Since $(x_n)$ is Cauchy,

$$\exists H \in \mathbb{N} \ni n, m \geq H \implies |x_n - x_m| < \delta.$$ 

But then

$$n, m \geq H \implies |f(x_n) - f(x_m)| < \epsilon.$$ 

Thus $(f(x_n))$ is Cauchy. \qed

**Example.**

(9) $g(x) = \frac{1}{x^2}$ is not u.c. on $(0, 1)$.

**Proof.** $(\frac{1}{n}) \to 0$ on $(0, 1)$, so is Cauchy on $(0, 1)$, but

$$\left(\frac{1}{x_n^2}\right) = \left(\frac{1}{(\frac{1}{n})^2}\right) = (n^2)$$ 

diverges, so is not Cauchy. Thus $g(x)$ is not u.c. on $(0, 1)$. \qed
Theorem (5.4.8 — Continuous Extension Theorem). $f$ is u.c. on $(a, b)$ if and only if it can be defined at the endpoints $a$ and $b$ implies the extended function $\overline{f}$ is continuous on $[a, b]$.

Proof.

$(\iff)$ $\overline{f}$ is continuous on $[a, b]$ implies $f$ is u.c. on $[a, b]$ implies $f$ is u.c. on $(a, b)$ by restriction.

$(\implies)$ Suppose $f$ is u.c. on $(a, b)$.

[We extend $f$ to $a$. Extending $f$ to $b$ is similar.]

Let $(x_n)$ be a sequence in $(a, b)$ implies $\lim(x_n) = a$.

Then $(x_n)$ is Cauchy implies $(f(x_n))$ is Cauchy implies

$\exists L \in \mathbb{R}$ implies $\lim(f(x_n)) = L$.

Suppose $(u_n)$ is any other sequence in $A$ implies $\lim(u_n) = a$.

Then $\lim(u_n - x_n) = a - a = 0$.

Since $f$ is u.c. on $(a, b)$,

$$
\lim_{x \to a} (f(u_n)) = \lim_{x \to a} (f(u_n) - f(x_n) + f(x_n)) =
\lim_{x \to a} (f(u_n) - f(x_n)) + \lim_{x \to a} (f(x_n)) = 0 + L = L.
$$

Then $\lim_{x \to a} f(x) = L$ by the sequential criterion for limits.

Defining $f(a) = L$ gives a continuous extension.
EXAMPLE.

(10) \( f(x) = \frac{\sin x}{x} \) is u.c. on \((0, b)\) \(\forall b > 0\).

PROOF.

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to b} \frac{\sin x}{x} = \frac{\sin b}{b}.
\]

Define \( f(0) = 1 \) and \( f(b) = \frac{\sin b}{b} \).

\(\square\)

(11) \( g(x) = \sin \left( \frac{1}{x} \right) \) is not u.c. on \((0, 1)\) since \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE.

**Homework**

Pages 148-49 # 1, 3b

(E1) Show \( f(x) = x^3 \) is u.c. on \((0, 2)\)

(E2) Show \( f(x) = \sin(x^2) \) is not u.c. on \(\mathbb{R} \) (Hint: use 5.4.2(iii))

(E3) Show \( f(x) = \sqrt{x} \) is u.c. on \([0, \infty)\).

(E4) Is \( f(x) = \frac{1}{x^3} \) u.c. on \((0, 1)\)? Prove.