Today’s Objectives:

Students will be able to:

1. Analyze the kinematics of a rigid body undergoing planar translation or rotation about a fixed axis.

In-Class Activities:

- Check Homework
- Reading Quiz
- Applications
- Types of Rigid-Body Motion
- Planar Translation
- Rotation About a Fixed Axis
- Concept Quiz
- Group Problem Solving
- Attention Quiz
READING QUIZ

1. If a rigid body is in translation only, the velocity at points A and B on the rigid body _______.
   A) are usually different
   B) are always the same
   C) depend on their position
   D) depend on their relative position

2. If a rigid body is rotating with a constant angular velocity about a fixed axis, the velocity vector at point P is _______.
   A) \( \omega \times r_p \)  
   B) \( r_p \times \omega \)
   C) \( \frac{dr_p}{dt} \)
   D) All of the above.
Passengers on this amusement ride are subjected to curvilinear translation since the vehicle moves in a circular path but they always remains upright.

If the angular motion of the rotating arms is known, how can we determine the velocity and acceleration experienced by the passengers? Why would we want to know these values? Does each passenger feel the same acceleration?
APPLICATIONS (continued)

Gears, pulleys and cams, which rotate about fixed axes, are often used in machinery to generate motion and transmit forces. The angular motion of these components must be understood to properly design the system.

To do this design, we need to relate the angular motions of contacting bodies that rotate about different fixed axes. How is this different than the analyses we did in earlier chapters?
There are cases where an object cannot be treated as a particle. In these cases the size or shape of the body must be considered. Rotation of the body about its center of mass requires a different approach.

For example, in the design of gears, cams, and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.

We will now start to study rigid body motion. The analysis will be limited to planar motion.

A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.
There are three types of planar rigid body motion.

- Path of rectilinear translation
- Path of curvilinear translation
- Rotation about a fixed axis
- General plane motion
Translation: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called rectilinear translation. When the paths of motion are curved lines, the motion is called curvilinear translation.
Rotation about a fixed axis: In this case, all the particles of the body, except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.

General plane motion: In this case, the body undergoes both translation and rotation. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.
An example of bodies undergoing the three types of motion is shown in this mechanism.

The wheel and crank undergo rotation about a fixed axis. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.

The piston undergoes rectilinear translation since it is constrained to slide in a straight line.

The connecting rod undergoes curvilinear translation, since it will remain horizontal as it moves along a circular path.

The connecting rod undergoes general plane motion, as it will both translate and rotate.
RIGID-BODY MOTION: TRANSLATION
(Section 16.2)

The positions of two points A and B on a translating body can be related by

\[ \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \]

where \( \mathbf{r}_A \) & \( \mathbf{r}_B \) are the absolute position vectors defined from the fixed x-y coordinate system, and \( \mathbf{r}_{B/A} \) is the relative-position vector between B and A.

The velocity at B is \( \mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt} \).

Now \( \frac{d\mathbf{r}_{B/A}}{dt} = 0 \) since \( \mathbf{r}_{B/A} \) is constant. So, \( \mathbf{v}_B = \mathbf{v}_A \), and by following similar logic, \( \mathbf{a}_B = \mathbf{a}_A \).

Note, all points in a rigid body subjected to translation move with the same velocity and acceleration.
RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS
(Section 16.3)

When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by \( \theta \).

The change in angular position, \( d\theta \), is called the angular displacement, with units of either radians or revolutions. They are related by

\[
1 \text{ revolution} = (2\pi) \text{ radians}
\]

Angular velocity, \( \omega \), is obtained by taking the time derivative of angular displacement:

\[
\omega = \frac{d\theta}{dt} \text{ (rad/s)}
\]

Similarly, angular acceleration is

\[
\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \quad \text{or} \quad \alpha = \omega \left( \frac{d\omega}{d\theta} \right)
\]

rad/s\(^2\)
If the angular acceleration of the body is constant, $\alpha = \alpha_C$, the equations for angular velocity and acceleration can be integrated to yield the set of algebraic equations below.

$$\omega = \omega_0 + \alpha_C t$$
$$\theta = \theta_0 + \omega_0 t + 0.5 \alpha_C t^2$$
$$\omega^2 = (\omega_0)^2 + 2\alpha_C (\theta - \theta_0)$$

$\theta_0$ and $\omega_0$ are the initial values of the body’s angular position and angular velocity. Note these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.
The magnitude of the velocity of P is equal to \( \omega r \) (the text provides the derivation). The velocity’s direction is tangent to the circular path of P.

In the vector formulation, the magnitude and direction of \( \mathbf{v} \) can be determined from the cross product of \( \omega \) and \( \mathbf{r}_p \). Here \( \mathbf{r}_p \) is a vector from any point on the axis of rotation to P.

\[
\mathbf{v} = \omega \times \mathbf{r}_p = \omega \times \mathbf{r}
\]

The direction of \( \mathbf{v} \) is determined by the right-hand rule.
RIGID-BODY ROTATION: ACCELERATION OF POINT P

The acceleration of P is expressed in terms of its normal \((a_n)\) and tangential \((a_t)\) components. In scalar form, these are
\[ a_t = \alpha \, r \] and \[ a_n = \omega^2 \, r. \]

The tangential component, \(a_t\), represents the time rate of change in the velocity's magnitude. It is directed tangent to the path of motion.

The normal component, \(a_n\), represents the time rate of change in the velocity’s direction. It is directed toward the center of the circular path.
Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\omega}{dt} \times \mathbf{r}_P + \omega \times \frac{d\mathbf{r}_P}{dt} \]

\[ = \alpha \times \mathbf{r}_P + \omega \times (\omega \times \mathbf{r}_P) \]

It can be shown that this equation reduces to

\[ \mathbf{a} = \alpha \times \mathbf{r} - \omega^2 \mathbf{r} = \mathbf{a}_t + \mathbf{a}_n \]

The magnitude of the acceleration vector is

\[ a = \sqrt{(a_t)^2 + (a_n)^2} \]
ROTATION ABOUT A FIXED AXIS: PROCEDURE

• Establish a **sign convention** along the axis of rotation.

• If a relationship is known between any **two** of the variables ($\alpha$, $\omega$, $\theta$, or $t$), the other variables can be determined from the equations:

$$\omega = \frac{d\theta}{dt} \quad \alpha = \frac{d\omega}{dt} \quad \alpha \ d\theta = \omega \ d\omega$$

• If $\alpha$ is **constant**, use the equations for constant angular acceleration.

• To determine the **motion of a point**, the scalar equations $v = \omega \ r$, $a_t = \alpha \ r$, $a_n = \omega^2 r$, and $a = \sqrt{(a_t)^2 + (a_n)^2}$ can be used.

• Alternatively, the **vector** form of the equations can be used (with $i, j, k$ components).

$$v = \omega \times r_P = \omega \times r$$

$$a = a_t + a_n = \alpha \times r_P + \omega \times (\omega \times r_P) = \alpha \times r - \omega^2 r$$
EXAMPLE

**Given:** The motor gives the blade an angular acceleration \( \alpha = 20 \, e^{-0.6t} \, \text{rad/s}^2 \), where \( t \) is in seconds. The initial conditions are that when \( t = 0 \), the blade is at rest.

**Find:** The velocity and acceleration of the tip P of one of the blades when \( t = 3 \, \text{s} \). How many revolutions has the blade turned in 3 s?

**Plan:**
1) Determine the angular velocity and displacement of the blade using kinematics of angular motion.
2) The magnitudes of the velocity and acceleration of point P can be determined from the scalar equations of motion for a point on a rotating body. Why scalar?
EXAMPLE (continued)

Solution:

1) Since the angular acceleration is given as a function of time, \( \alpha = 20 \ e^{-0.6t} \ \text{rad/s}^2 \), the angular velocity and displacement can be found by integration.

\[
\omega = \int \alpha \ dt = 20 \int e^{-0.6t} \ dt
\]

\[
\omega = \frac{20}{(-0.6)} \ e^{-0.6t}
\]

\( \Rightarrow \) when \( t = 3 \ \text{s} \),

\( \omega = -5.510 \ \text{rad/s} \)

Angular displacement

\[
\theta = \int \omega \ dt
\]

\[
\theta = \frac{20}{(-0.6)} \int e^{-0.6t} \ dt = \frac{20}{(-0.6)^2} \ e^{-0.6t}
\]

\( \Rightarrow \) when \( t = 3 \ \text{s} \),

\( \theta = 9.183 \ \text{rad} \)

\( = 1.46 \ \text{rev.} \)

Also, when \( t = 3 \ \text{s} \), \( \alpha = 20 \ e^{-0.6(3)} = 3.306 \ \text{rad/s}^2 \)
EXAMPLE (continued)

2) The velocity of point P on the fan, at a radius of 1.75 ft, is determined as

\[ v_P = \omega r = (5.510)(1.75) = 9.64 \text{ ft/s} \]

The normal and tangential components of acceleration of point P are calculated as

\[ a_n = (\omega)^2 r = (5.510)^2 (1.75) = 53.13 \text{ ft/s}^2 \]
\[ a_t = \alpha r = (3.306)(1.75) = 5.786 \text{ ft/s}^2 \]

The magnitude of the acceleration of P is determined by

\[ a_P = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(53.13)^2 + (5.786)^2} = 53.4 \text{ ft/s}^2 \]
1. A disk is rotating at 4 rad/s. If it is subjected to a constant angular acceleration of 2 rad/s², determine the acceleration at B.

   A) \((4 \mathbf{i} + 32 \mathbf{j}) \text{ ft/s}^2\)   B) \((4 \mathbf{i} - 32 \mathbf{j}) \text{ ft/s}^2\)
   C) \((-4 \mathbf{i} + 32 \mathbf{j}) \text{ ft/s}^2\)   D) \((-4 \mathbf{i} - 32 \mathbf{j}) \text{ ft/s}^2\)

2. A Frisbee is thrown and curves to the right. It is experiencing

   A) rectilinear translation.  B) curvilinear translation.
   C) pure rotation.  D) general plane motion.
GROUP PROBLEM SOLVING

**Given:** Starting from rest when gear A is given a constant angular acceleration, $\alpha_A = 4.5 \text{ rad/s}^2$. The cord is wrapped around pulley D which is rigidly attached to gear B.

**Find:** The velocity of cylinder C and the distance it travels in 3 seconds.

**Plan:**

1) The angular acceleration of gear B (and pulley D) can be related to $\alpha_A$.
2) The acceleration of cylinder C can be determined by using the equations for motion of a point on a rotating body since $(a_t)_D$ at point P is the same as $a_c$.
3) The velocity and distance of C can be found by using the constant acceleration equations.
GROUP PROBLEM SOLVING
(continued)

Solution:

1) Gear A and B will have the same speed and tangential component of acceleration at the point where they mesh. Thus,

\[ a_t = \alpha_A r_A = \alpha_B r_B \quad \Rightarrow \quad (4.5)(75) = \alpha_B (225) \quad \Rightarrow \quad \alpha_B = 1.5 \text{ rad/s}^2 \]

Since gear B and pulley D turn together, \( \alpha_D = \alpha_B = 1.5 \text{ rad/s}^2 \)

2) Assuming the cord attached to pulley D is inextensible and does not slip, the velocity and acceleration of cylinder C will be the same as the velocity and tangential component of acceleration along the pulley D:

\[ a_C = (a_t)_D = \alpha_D r_D = (1.5)(0.125) = 0.1875 \text{ m/s}^2 \]
GROUP PROBLEM SOLVING
(continued)

3) Since $\alpha_A$ is constant, $\alpha_D$ and $a_C$ will be constant. The constant acceleration equation for rectilinear motion can be used to determine the velocity and displacement of cylinder C when $t = 3$ s ($s_0 = v_0 = 0$):

$$v_c = v_0 + a_C t = 0 + 0.1875 \times 3 = 0.563 \text{ m/s}$$

$$s_c = s_0 + v_0 t + (0.5) a_C t^2 = 0 + 0 + (0.5) 0.1875 \times (3)^2 = 0.844 \text{ m}$$
ATTENTION QUIZ

1. The fan blades suddenly experience an angular acceleration of 2 rad/s\(^2\). If the blades are rotating with an initial angular velocity of 4 rad/s, determine the speed of point P when the blades have turned 2 revolutions (when \(\omega = 8.14\) rad/s).

   A) 14.2 ft/s  B) 17.7 ft/s  
   C) 23.1 ft/s  D) 26.7 ft/s

2. Determine the magnitude of the acceleration at P when the blades have turned the 2 revolutions.

   A) 0 ft/s\(^2\)  B) 3.5 ft/s\(^2\)  
   C) 115.95 ft/s\(^2\)  D) 116 ft/s\(^2\)
End of the Lecture

Let Learning Continue