EQUATIONS OF MOTION: NORMAL AND TANGENTIAL COORDINATES

Today’s Objectives:
Students will be able to:
1. Apply the equation of motion using normal and tangential coordinates.

In-Class Activities:
- Check Homework
- Reading Quiz
- Applications
- Equation of Motion in n-t Coordinates
- Concept Quiz
- Group Problem Solving
- Attention Quiz
READING QUIZ

1. The “normal” component of the equation of motion is written as $\sum F_n = ma_n$, where $\sum F_n$ is referred to as the _______.
   
   A) impulse  B) centripetal force  
   C) tangential force  D) inertia force  

2. The positive $n$ direction of the normal and tangential coordinates is ____________.
   
   A) normal to the tangential component  
   B) always directed toward the center of curvature  
   C) normal to the bi-normal component  
   D) All of the above.
Applications

Race tracks are often banked in the turns to reduce the frictional forces required to keep the cars from sliding up to the outer rail at high speeds.

If the car’s maximum velocity and a minimum coefficient of friction between the tires and track are specified, how can we determine the minimum banking angle ($\theta$) required to prevent the car from sliding up the track?
The picture shows a ride at the amusement park. The hydraulically-powered arms turn at a constant rate, which creates a centrifugal force on the riders.

We need to determine the smallest angular velocity of the cars A and B so that the passengers do not lose contact with the seat. What parameters do we need for this calculation?
Satellites are held in orbit around the earth by using the earth’s gravitational pull as the centripetal force – the force acting to change the direction of the satellite’s velocity.

Knowing the radius of orbit of the satellite, we need to determine the required speed of the satellite to maintain this orbit. What equation governs this situation?
NORMAL & TANGENTIAL COORDINATES  
(Section 13.5)  

When a particle moves along a curved path, it may be more convenient to write the equation of motion in terms of normal and tangential coordinates.

The normal direction (n) *always* points toward the path’s center of curvature. In a circle, the center of curvature is the center of the circle.

The tangential direction (t) is tangent to the path, usually set as positive in the direction of motion of the particle.
EQUATIONS OF MOTION

Since the equation of motion is a vector equation, \( \sum F = ma \), it may be written in terms of the \( n \) & \( t \) coordinates as

\[
\sum F_t u_t + \sum F_n u_n + \sum F_b u_b = ma_t + ma_n
\]

Here \( \sum F_t \) & \( \sum F_n \) are the sums of the force components acting in the \( t \) & \( n \) directions, respectively.

This vector equation will be satisfied provided the individual components on each side of the equation are equal, resulting in the two scalar equations: \( \sum F_t = ma_t \) and \( \sum F_n = ma_n \).

Since there is no motion in the binormal (b) direction, we can also write \( \sum F_b = 0 \).
NORMAL AND TANGENTIAL ACCELERATIONS

The **tangential acceleration**, $a_t = \frac{dv}{dt}$, represents the time rate of change in the magnitude of the velocity. Depending on the direction of $\Sigma F_t$, the particle’s speed will either be increasing or decreasing.

The **normal acceleration**, $a_n = \frac{v^2}{\rho}$, represents the time rate of change in the direction of the velocity vector. Remember, $a_n$ always acts toward the path’s center of curvature. Thus, $\Sigma F_n$ will always be directed toward the center of the path.

Recall, if the path of motion is defined as $y = f(x)$, the **radius of curvature** at any point can be obtained from

$$\rho = \frac{[1 + \left(\frac{dy}{dx}\right)^2]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$
SOLVING PROBLEMS WITH n-t COORDINATES

• Use n-t coordinates when a particle is moving along a known, curved path.

• Establish the n-t coordinate system on the particle.

• Draw free-body and kinetic diagrams of the particle. The normal acceleration \( (a_n) \) always acts “inward” (the positive n-direction). The tangential acceleration \( (a_t) \) may act in either the positive or negative t direction.

• Apply the equations of motion in scalar form and solve.

• It may be necessary to employ the kinematic relations:

\[
a_t = \frac{dv}{dt} = v \frac{dv}{ds} \quad a_n = \frac{v^2}{\rho}
\]
EXAMPLE

Given: At the instant $\theta = 45^\circ$, the boy with a mass of 75 kg, moves a speed of 6 m/s, which is increasing at 0.5 m/s$^2$. Neglect his size and the mass of the seat and cords. The seat is pin connected to the frame BC.

Find: Horizontal and vertical reactions of the seat on the boy.

Plan:

1) Since the problem involves a curved path and requires finding the force perpendicular to the path, use n-t coordinates. Draw the boy’s free-body and kinetic diagrams.
2) Apply the equation of motion in the n-t directions.
EXAMPLE
(continued)

Solution:

1) The n-t coordinate system can be established on the boy at angle $45^\circ$. Approximating the boy and seat together as a particle, the free-body and kinetic diagrams can be drawn.
EXAMPLE (continued)

2) Apply the equations of motion in the n-t directions.

(a) \( \sum F_n = m a_n \Rightarrow -R_x \cos 45^\circ - R_y \sin 45^\circ + W \sin 45^\circ = ma_n \)

Using \( a_n = v^2/\rho = 6^2/10 \), \( W = 75(9.81) \) N, and \( m = 75 \) kg,

we get: \( -R_x \cos 45^\circ - R_y \sin 45^\circ + 520.3 = (75)(6^2/10) \) \hspace{1cm} (1)

(b) \( \sum F_t = m a_t \Rightarrow -R_x \sin 45^\circ + R_y \cos 45^\circ - W \cos 45^\circ = ma_t \)

we get: \( -R_x \sin 45^\circ + R_y \cos 45^\circ - 520.3 = 75 (0.5) \) \hspace{1cm} (2)

Using equations (1) and (2), solve for \( R_x \), \( R_y \).

\[ R_x = -217 \text{ N}, \quad R_y = 572 \text{ N} \]
CONCEPT QUIZ

1. A 10 kg sack slides down a smooth surface. If the normal force on the surface at the flat spot, A, is 98.1 N (↑), the radius of curvature is _____.
   A) 0.2 m    B) 0.4 m
   C) 1.0 m    D) None of the above.

2. A 20 lb block is moving along a smooth surface. If the normal force on the surface at A is 10 lb, the velocity is ________.
   A) 7.6 ft/s    B) 9.6 ft/s
   C) 10.6 ft/s   D) 12.6 ft/s
GROUP PROBLEM SOLVING

**Given:** A 800 kg car is traveling over the hill having the shape of a parabola. When it is at point A, it is traveling at 9 m/s and increasing its speed at 3 \( m/s^2 \).

**Find:** The resultant normal force and resultant frictional force exerted on the road at point A.

**Plan:**
1) Treat the car as a particle. Draw the free-body and kinetic diagrams.
2) Apply the equations of motion in the n-t directions.
3) Use calculus to determine the slope and radius of curvature of the path at point A.
GROUP PROBLEM SOLVING
(continued)

Solution:

1) The n-t coordinate system can be established on the car at point A. Treat the car as a particle and draw the free-body and kinetic diagrams:

\[
\begin{align*}
W &= mg = \text{weight of car} \\
N &= \text{resultant normal force on road} \\
F &= \text{resultant friction force on road}
\end{align*}
\]
GROUP PROBLEM SOLVING
(continued)

2) Apply the equations of motion in the n-t directions:

\[ \sum F_n = ma_n \Rightarrow W \cos \theta - N = ma_n \]

Using \( W = mg \) and \( a_n = v^2/\rho = (9)^2/\rho \)

\[ \Rightarrow (800)(9.81) \cos \theta - N = (800) (81/\rho) \]

\[ \Rightarrow N = 7848 \cos \theta - 64800/\rho \] (1)

\[ \sum F_t = ma_t \Rightarrow W \sin \theta - F = ma_t \]

Using \( W = mg \) and \( a_t = 3 \text{ m/s}^2 \) (given)

\[ \Rightarrow (800)(9.81) \sin \theta - F = (800) \] (3)

\[ \Rightarrow F = 7848 \sin \theta - 2400 \] (2)
GROUP PROBLEM SOLVING
(continued)

3) **Determine** $\rho$ by differentiating $y = f(x)$ at $x = 80$ m:

\[
y = 20(1 - x^2/6400) \Rightarrow \frac{dy}{dx} = \frac{-40}{6400} x
\]

\[
\Rightarrow \frac{d^2y}{dx^2} = \frac{-40}{6400}
\]

\[
\tan q = \frac{dy}{dx}
\]

\[
q = \tan^{-1}\left(\frac{dy}{dx}\right) = \tan^{-1}\left(-0.5\right) = 26.6^\circ
\]

\[
\rho \bigg|_{x = 80 \text{ m}} = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1 + (-0.5)^2]^{3/2}}{0.00625} = 223.6 \text{ m}
\]

**Determine** $\theta$ from the slope of the curve at A:

\[
\tan \theta = \frac{dy}{dx} \bigg|_{x = 80 \text{ m}}
\]

\[
\theta = \left|\tan^{-1}\left(\frac{dy}{dx}\right)\right| = \left|\tan^{-1}\left(-0.5\right)\right| = 26.6^\circ
\]
GROUP PROBLEM SOLVING
(continued)

From Eq.(1): \( N = 7848 \cos \theta - 64800 / \rho \)
\[ \quad = 7848 \cos (26.6^\circ) - 64800 / 223.6 = 6728 \text{ N} \]

From Eq.(2): \( F = 7848 \sin \theta - 2400 \)
\[ \quad = 7848 \sin (26.6^\circ) - 2400 = 1114 \text{ N} \]
ATTENTION QUIZ

1. The tangential acceleration of an object
   A) represents the rate of change of the velocity vector’s direction.
   B) represents the rate of change in the magnitude of the velocity.
   C) is a function of the radius of curvature.
   D) Both B and C.

2. The block has a mass of 20 kg and a speed of \( v = 30 \, \text{m/s} \) at the instant it is at its lowest point. Determine the tension in the cord at this instant.
   A) 1596 N       B) 1796 N
   C) 1996 N       D) 2196 N
End of the Lecture

Let Learning Continue