PART I. Do not use a calculator on this part.

Complete the following statements. (12 points)

1. Is the following statement true or false? If a function is continuous at \( x = 6 \), then it is also differentiable at \( x = 6 \).
   
   \text{FALSE} 

2. A function \( h \) is continuous at \( x = b \) if \( \lim_{x \to b} h(x) = h(b) \).

3. Let \( f, g, h \) be functions such that \( f(x) \leq g(x) \leq h(x) \) for all \( x \) in an interval \((-5, 5)\). If 
   \[ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = 2, \text{ then } \lim_{x \to a} g(x) = 2, \] 
   according to the \textbf{Squeeze} \textbf{theorem}.

4. If \( x < 4 \), then \( |x - 4| = -(x - 4) \). (Write the answer in terms of \( x \) without using absolute value bars.)

5. The derivative of a function \( f \) is the function \( f' \) that is defined by 
   \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

6. Determine \( \lim_{x \to 0} \frac{2 - 9x^2}{3x^2 + x - 1} \).
   
   \[ \lim_{x \to 0} \frac{2 - 9x^2}{3x^2 + x - 1} = \lim_{x \to 0} \frac{2x^2 - 9}{3 + \frac{1}{x} - x^2} = -\frac{9}{3} = -3 \] 

7. Determine \( \lim_{x \to 2} \frac{x}{4 - x^2} \).
   
   As \( x \to 2 \), \( 4 - x^2 \to 0 \). When \(-2 < x < 2\), \((2-x)(2+x) > 0\).
   
   So, \( \lim_{x \to 2} \frac{x}{4 - x^2} = +\infty \)

8. Determine all vertical and horizontal asymptotes of \( f(x) = \frac{3x^2 + 1}{x^2 + x - 6} \).
   
   \( f(x) = \frac{3x^2 + 1}{(x-2)(x+3)} \)
   
   Vertical asymptotes: \( x = 2 \) and \( x = -3 \)

   \[ \lim_{x \to \pm \infty} \frac{3x^2 + 1}{x^2 + x - 6} = \lim_{x \to \infty} \frac{3 + \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{6}{x^2}} = 3 \]

   \( y = 3 \) is the horizontal asymptote.
9. Let \( f(x) = \begin{cases} x^2 - 5b & \text{if } x < 0 \\
9 + \cos x & \text{if } x > 0 \end{cases} \), where \( b \) denotes a constant. What would make \( f \) continuous at \( x = 0 \)?

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 - 5b) = -5b
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (9 + \cos x) = 9 + \cos 0 = 9 + 1 = 10.
\]
Thus, \( \lim_{x \to 0} f(x) = 10 \) if \( -5b = 10 \); so \( b = -2 \).

The function \( f \) is continuous at \( x = 0 \) if

(i) \( b = -2 \), and

(ii) the discontinuity at \( x = 0 \) is removed by defining \( f(0) = 10 \).

10. Let \( f(x) = \sin^{-1} x \) and \( g(x) = x - 3 \).

(a) Find \((f \circ g)(x)\).

\[(f \circ g)(x) = f(g(x)) = \sin^{-1} g(x) = \sin^{-1} (x-3)\).

(b) Determine the interval on which \((f \circ g)(x)\) is continuous.

The function \( g \) is continuous on \((-\infty, \infty)\), and \( f \) is continuous on \((-1, 1)\).

Therefore, \( f \circ g \) is continuous at \( x \) if \(-1 < x - 3 < 1\).

So, \( f \circ g \) is continuous on \((2, 4)\).

11. Let \( f(x) = \frac{6}{x+1} \). Use the definition of the derivative (NOT the rules) to compute the derivative function \( f' \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6}{x+h+1} - \frac{6}{x+1}
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{6}{x+h+1} - \frac{6}{x+1} \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{6(x+1) - 6(x+h+1)}{(x+h+1)(x+1)} \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{6x+6-6x-6h-6}{(x+h+1)(x+1)} \right] = \lim_{h \to 0} \frac{1}{h} \left[ \frac{-6h}{(x+h+1)(x+1)} \right]
\]
\[
= \lim_{h \to 0} \frac{-6}{(x+h+1)(x+1)} = \frac{-6}{(x+1)^2}
\]

Thus, \( f'(x) = -\frac{6}{(x+1)^2} \)
12. Find the equation of the tangent line to \( f(x) = 3x^2 - 2 \) at \( x = -1 \).

When \( x = -1 \),
\[
y = f(-1) = 3(-1)^2 - 2 = 1
\]
Differentiating,
\[
f'(x) = 6x.
\]
So the slope of the tangent line at \( x = -1 \) is \( f'(-1) = 6(-1) = -6 \).

The equation of the tangent line is
\[
y - 1 = -6(x+1) \Rightarrow y = -6x - 5
\]

13. Use the rules to differentiate the following functions:

(a) \( y = 4x^4 - 5x \)

\[
y' = 16x^3 - 5
\]

(b) \( y = 2\sqrt{x} - \frac{3}{x^5} = 2x^{\frac{1}{2}} - 3x^{-5} \)

\[
y' = \frac{d}{dx} \left( 2x^{\frac{1}{2}} - 3x^{-5} \right) = 2 \left( \frac{1}{2} x^{-\frac{1}{2}} - 3(-5)x^{-6} \right)
\]
\[
= x^{-\frac{1}{2}} + 15x^{-6}. \quad \therefore \quad y' = \frac{1}{\sqrt{x}} + \frac{15}{x^6}
\]

(c) \( y = \frac{5x-4}{x+1} \) By the Quotient Rule,

\[
y' = \frac{(x+1) \frac{d}{dx}(5x-4) - (5x-4) \frac{d}{dx}(x+1)}{(x+1)^2}
\]
\[
= \frac{5x + 5 - 5x + 4}{(x+1)^2} \Rightarrow y' = \frac{9}{(x+1)^2}
\]

14. Use the rules to find the velocity and acceleration functions of an object whose position function is

\( s(t) = -16t^2 + 50t \).

**Velocity:**
\[
y = \frac{dy}{dt} = \frac{d}{dt} (-16t^2 + 50t) = -32t + 50 \Rightarrow y = -32t + 50
\]

**Acceleration:**
\[
a = \frac{dv}{dt} = \frac{d}{dt} (-32t + 50) = -32 \Rightarrow a = -32
\]
PART II. (14 points) Use your calculator to work the following problems.

1. Let \( f(x) = \sqrt{1 + \tan x} \). Numerically estimate the derivative \( f'(1) \) to three decimal places. Do this by using a difference quotient to numerically estimate the left-hand and right-hand derivatives at \( x = 1 \). Include a table showing at least 3 estimates of the left-hand derivative and at least 3 estimates of the right-hand derivative.

   On the TI-89, define the function:
   
   Define \( f(x) = \sqrt{1 + \tan(x)} \).

   Then define the difference quotient: \( d(x) = (f(x) - f(1)) / (x-1) \).

   **Left-hand derivative**:

   \[
   \begin{array}{c|c}
   x & d(x) \\
   \hline
   0.5 & 0.71137 \\
   0.9 & 0.95807 \\
   0.99 & 1.0581 \\
   0.999 & 1.06941 \\
   0.9999 & 1.0709 \\
   \end{array}
   \]

   **Right-hand derivative**:

   \[
   \begin{array}{c|c}
   x & d(x) \\
   \hline
   1.5 & 4.5737 \\
   1.01 & 1.2266 \\
   1.001 & 1.0843 \\
   1.0001 & 1.0723 \\
   1.00001 & 1.0711 \\
   \end{array}
   \]

   From the table it appears that 
   \( f'(1) = 1.071 \) to three decimal places.
2. The function \( s(t) = 2t^3 + 1 \) represents the position in meters of an object at time \( t \) seconds. Compute the average velocity of the object between (a) \( t = 1.99 \) and \( t = 2 \), (b) \( t = 1.999 \) and \( t = 2 \), (c) \( t = 2.01 \) and \( t = 2 \), (d) \( t = 2.001 \) and \( t = 2 \). Estimate the instantaneous velocity at \( t = 2 \) seconds. What are the units of velocity?

The average velocity between times \( t \) and \( \bar{t} \) is \( \frac{s(t) - s(\bar{t})}{t - \bar{t}} \).

As we showed several times in class, the calculations can be carried out quickly on the TI-89 by defining the functions

- Define \( s(t) = 2t^3 + 1 \)
- Define \( d(t) = (s(t) - s(2))/(t - 2) \).

(a) \( d(1.99) = 23.8802 \)
(b) \( d(1.999) = 23.488 \)
(c) \( d(2.01) = 24.1202 \)
(d) \( d(2.001) = 24.012 \)

From (a) - (d), we estimate the instantaneous velocity at \( t = 2 \) is \( 24 \text{ m/s} \).

The units of velocity are \( \text{meters per second} \).