CHAPTER 6

Mathematics of Finance

1. Solving Exponential Equations

Method for solving equations of the form \( y = ab^x \).

\[
y = ab^x
\]

(1) Divide both sides by \( a \).

\[
\frac{y}{a} = b^x
\]

(2) Take the log or ln of both sides.

\[
\log \frac{y}{a} = \log b^x \quad \text{or} \quad \ln \frac{y}{a} = \ln b^x
\]

(3) Bring the \( x \) down in front of log or ln.

\[
\log \frac{y}{a} = x \log b \quad \text{or} \quad \ln \frac{y}{a} = x \ln b
\]

(4) Divide both sides by \( \log b \) or \( \ln b \), respectively.

\[
x = \frac{\log \frac{y}{a}}{\log b} \quad \text{or} \quad x = \frac{\ln \frac{y}{a}}{\ln b}
\]

Problem (Page 389 #4).

\[
5(0.5)^x = 0.125
\]

\[
0.5^x = 0.025
\]

\[
\ln 0.5^x = \ln 0.025
\]

\[
x \ln 0.5 = \ln 0.025
\]

\[
x = \frac{\ln 0.025}{\ln 0.5} \approx 5.32
\]
**Problem** (Page 389 #14).

\[ t = \# \text{ of years since the end of 1980.} \]

\[ M = \text{annual per capita milk beverage consumption in gallons.} \]

\[ M(t) = 27.76(0.9914)^t \]

When is \( M(t) = 20? \) Solve

\[
\begin{align*}
20 &= 27.76(0.9914)^t \\
\frac{20}{27.76} &= 0.9914^t \\
\ln \frac{20}{27.76} &= \ln 0.9914^t \\
\ln 20 - \ln 27.76 &= t \ln 0.9914 \\
\end{align*}
\]

\[ t = \frac{\ln 20 - \ln 27.76}{\ln 0.9914} \approx 37.96 \]

The average person will drink 20 gallons of milk in 2018.
Method for solving equations of the form $ab^x = cd^x$.

(1) Divide both sides by $a$. 

$$ab^x = cd^x$$

$$b^x = \frac{cd^x}{a}$$

(2) Divide both sides by $d^x$. 

$$\frac{b^x}{d^x} = \frac{c}{a}$$

(3) Rewrite $\frac{b^x}{d^x}$ as $\left(\frac{b}{d}\right)^x$. 

$$\left(\frac{b}{d}\right)^x = \frac{c}{a}$$

(4) Solve using previous rule. 

$$\ln\left(\frac{b}{d}\right)^x = \ln\left(\frac{c}{a}\right)$$

$$x \ln\left(\frac{b}{d}\right) = \ln\left(\frac{c}{a}\right)$$

$$x = \frac{\ln\left(\frac{c}{a}\right)}{\ln\left(\frac{b}{d}\right)}$$
Example.

\[ 4 \cdot 5^x = 7 \cdot 2^x \]
\[ 5^x = \frac{7 \cdot 2^x}{4} \]
\[ \frac{5^x}{2^x} = \frac{7}{4} \]
\[ \left( \frac{5}{2} \right)^x = \frac{7}{4} \]
\[ \ln \left( \frac{5}{2} \right)^x = \ln \left( \frac{7}{4} \right) \]
\[ x \ln \left( \frac{5}{2} \right) = \ln \left( \frac{7}{4} \right) \]
\[ x = \frac{\ln \left( \frac{7}{4} \right)}{\ln \left( \frac{5}{2} \right)} = \frac{\ln 7 - \ln 4}{\ln 5 - \ln 2} \approx .61 \]
**Problem** (Page 389 #16).

$t = \#$ of years since the end of 1980.

$M(t) =$ per capita gallons of milk consumed in year $t$

$$M(t) = 27.76(0.9914)^t$$

$W(t) =$ per capita gallons of bottled water consumed in year $t$

$$W(t) = 2.593(1.106)^t$$

When is $M(t) = W(t)$?

$$27.76(0.9914)^t = 2.593(1.106)^t$$

$$\frac{0.9914^t}{1.106^t} = \frac{2.593}{27.76}$$

$$\left(\frac{0.9914}{1.106}\right)^t = \frac{2.593}{27.76}$$

$$\ln \left(\frac{0.9914}{1.106}\right)^t = \ln \frac{2.593}{27.76}$$

$$t \ln \left(\frac{0.9914}{1.106}\right) = \ln \frac{2.593}{27.76}$$

$$t = \frac{\ln \frac{2.593}{27.76}}{\ln \frac{0.9914}{1.106}} \approx 21.67$$

Annual per capita consumption of water will surpass that of milk about two-thirds of the way through 2002.
**Problem** (Page 391 #28).

\(t = \text{time in years.}\)

1st account: \(y = 500(1 + .092)^t\) or \(y = 500(1.092)^t\).

2nd account: \(y = 1000(1 + .053)^t\) or \(y = 1000(1.053)^t\).

On TI:

\[
\begin{align*}
Y_1 &= 500 \times 1.092 \times X \\
Y_2 &= 1000 \times 1.053 \times X
\end{align*}
\]

Set a window \([0, 25] \times [0, 3500]\).

Use **2nd/CALC/5:intersect** to find that the graphs meet at the point \((19.06, 2675.92)\).

Thus it takes a little over 19 years for the two accounts to have the same amount of money, $2675.92.

**Problem** (Page 391 #32).

Does

\(3^x = -x^2 + 4x - 2\)

have a solution.

On TI, graph:

\[
\begin{align*}
Y_1 &= 3 \times X \\
Y_2 &= -X \times 2 + 4 \times X - 2
\end{align*}
\]

We see the graphs never meet, so there is no solution.
Problem (Page 391 #34).

\[5(2^{x+1}) = 2^{2x-1}\]

\[5 = \frac{2^{2x-1}}{2^{x+1}}\]

\[5 = 2^{(2x-1)-(x+1)}\]

\[5 = 2^{x-2}\]

\[\ln 5 = \ln 2^{x-2}\]

\[\ln 5 = (x-2)\ln 2\]

\[\frac{\ln 5}{\ln 2} = x - 2\]

\[x = 2 + \frac{\ln 5}{\ln 2} \approx 4.32\]

2. Simple and Compound Interest

What is interest?

It is a fee paid for using another’s money. It is a dollar amount.

Simple Interest — the basis for all interest.

\[I = Prt\]

\(I = \text{interest}\) (usually in dollars).

\(P = \text{principal}\) or \(\text{present value}\) of an investment.

\(r = \text{interest rate}\) (usually annual interest rate - we assume this unless told otherwise).

\(t = \text{time}\) (usually in years).

Note. \(r\) and \(t\) must agree in time units, e.g., dollars per year and years, dollars per month and months, etc.
Related formulas:

\[ P = \frac{I}{rt}, \quad r = \frac{I}{Pr}, \quad t = \frac{I}{Pr} \]

Simple Interest:
The **future value** \( A \) of an initial investment \( P \) earning a simple interest rate \( r \) (or \( 100r\% \)) is given by

\[ A = P + Prt = P(1 + rt) \]

where \( t \) is the number of years after the initial investment is made. The rate \( r \) is the decimal form of the percentage rate (e.g., .072 for 7.2%).

**Time line:**

<table>
<thead>
<tr>
<th>0</th>
<th>( P )</th>
<th>( t ) years</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Problem** (Page 402 #2).

\[ r = 4.25\% = .0425, \quad t = 5, \quad P = 1000 \]

\[ A = P(1 + rt) = 1000(1 + .0425(5)) = 1212.50 \]

\[ I = A - P = 1212.50 - 1000 = 212.50 \]

The CD will be worth $1212.50 at maturity with $212.50 coming from interest.

**Problem** (Page 402 #4).

Let \( A = 2P \) and \( t = 7 \).

\[ A = P(1 + rt) \]

\[ 2P = P(1 + 7r) \]

\[ 2 = 1 + 7r \]

\[ 1 = 7r \]

\[ r = \frac{1}{7} \approx .1429 = 14.29\% \]

The investor will need a 14.29\% simple interest rate.
Simple interest investments earn interest only on the initial amount invested.

\[ A = P + Pr\, t \]

Simple interest investments grow linearly with time.

Compound interest investments earn interest on the initial amount invested and any previously earned interest.

**Example.** Suppose $100 is invested at 4% (annual rate) compounded monthly. We use

\[ A = P(1 + rt) \]

for one month.

Use

\[ r = .04 \text{ and } t = \frac{1}{12} \quad \text{or} \quad r = \frac{.04}{12} \text{ and } t = 1 \]

Month 1: \[ A = 100 \left(1 + \frac{.04}{12}\right) \]

Month 2: \[ A = \left[100 \left(1 + \frac{.04}{12}\right)\right] \left(1 + \frac{.04}{12}\right) = 100 \left(1 + \frac{.04}{12}\right)^2 \]

Month 3: \[ A = \left[100 \left(1 + \frac{.04}{12}\right)^2\right] \left(1 + \frac{.04}{12}\right) = 100 \left(1 + \frac{.04}{12}\right)^3 \]

\[ \vdots \]

\[ \vdots \]
Compound Interest:

The future value \( A \) of an initial investment \( P \) earning a compound interest rate \( 100r \) percent is given by

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

where \( n \) is the number of times the interest is paid annually and \( t \) is the number of years after the initial investment is made. Again, \( r \) is the decimal form of the percentage rate.

\[
\begin{array}{cccc}
\text{P} & & & \text{A} \\
0 & 1 & 2 & \text{nt periods}
\end{array}
\]

Note that

\[ I = A - P. \]

The compounding frequency is the number of times the interest is paid each year.

<table>
<thead>
<tr>
<th>Interest is Compounded</th>
<th>Compounding Frequency ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>1</td>
</tr>
<tr>
<td>Semiannually</td>
<td>2</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4</td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
</tr>
<tr>
<td>Daily</td>
<td>365</td>
</tr>
</tbody>
</table>

Compound interest investments grow exponentially. \( b = 1 + \frac{r}{n} \) is the periodic growth factor and \( \frac{r}{n} \) is the periodic growth rate.

**Problem** (Page 402 #6).

\[
t = 5, \quad r = 4.69\% = .0469, \quad n = 12, \quad P = 1000
\]

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt} = 1000 \left( 1 + \frac{.0469}{12} \right)^{12(5)} = $1263.70
\]

\[
I = A - P = 1263.70 - 1000 = $263.70
\]
Annual Percentage Yield (APY)

The annual percentage yield is the percentage rate at which interest would need to be paid if interest were calculated and paid only once a year.

APY is used to compare investments.

**Theorem.** The annual percentage yield, APY, for an account earning \(100r\) percent compounded \(n\) times per year is

\[
APY = \left(1 + \frac{r}{n}\right)^n - 1.
\]

For a simple interest account,

\[
APY = r.
\]

**Note.**

\[
A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^n\right]^t = P b^t
\]

if \(r\) and \(n\) are known. Recall that \(b\) is the annual growth factor, and

\[
APY = b - 1
\]

is the annual growth rate of the investment.

**Problem** (Page 403 #12).

\[
r = 1.25\% = .0125, \quad n = 365
\]

\[
APY = \left(1 + \frac{.0125}{365}\right)^{365} - 1 \approx .01258 = 1.258\%
\]
Continuous Compound Interest

What happens as we make more and more compounding periods in a year? Suppose \( r = 0.05 \).

<table>
<thead>
<tr>
<th>Compounding Frequency</th>
<th>Number of times compounded yearly ((n))</th>
<th>APY (= \left(1 + \frac{r}{n}\right)^n - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>1</td>
<td>0.05000</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4</td>
<td>0.05095</td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
<td>0.05116</td>
</tr>
<tr>
<td>Daily</td>
<td>365</td>
<td>0.05127</td>
</tr>
<tr>
<td>Hourly</td>
<td>8760</td>
<td>0.05127</td>
</tr>
<tr>
<td>Every Minute</td>
<td>525,600</td>
<td>0.05127</td>
</tr>
<tr>
<td>Every Second</td>
<td>31,536,000</td>
<td>0.05127</td>
</tr>
</tbody>
</table>

A limit of sorts has been reached. Recall

\[
\left(1 + \frac{1}{n}\right)^n \to e \quad \text{as} \quad n \to \infty.
\]

Similarly,

\[
\left(1 + \frac{r}{n}\right)^n \to e^r \quad \text{as} \quad n \to \infty.
\]

For instance,

\[
\left(1 + \frac{0.05}{1,000,000}\right)^{1,000,000} = 1.051271095
\]

and

\[ e^{.05} = 1.051271096. \]
Continuous Compounding

For infinitely large $n$, we use continuous compounding.

**THEOREM.** The future value $A$ of an initial investment $P$ earning a continuous compound interest rate $100r$ percent is given by

$$A = Pe^{rt}$$

where $t$ is the number of years after the initial investment is made. Then

$$APY = e^r - 1.$$ 

**Problem** (Page 403 #22). 

We have 2.98% compounded continuously.

$$r = APY = e^{0.0298} - 1 \approx .0302 = 3.02\%$$

**Problem** (Page 403 #24).

We have a 4.75% annual rate.

$$e^r - 1 = .0475 \quad (APY)$$

$$e^r = 1.0475$$

$$\ln e^r = \ln 1.0475$$

$$r \ln e = \ln 1.0475$$

$$r = \ln 1.0475 \quad (\text{since } \ln e = 1)$$

$$r \approx .0464 = 4.64\% \text{ continuous rate.}$$
Problem (Page 403 #18).

Account 1: \( P = 2000, \quad r = .04, \quad \) continuous
\[
A = Pe^{rt} = 2000e^{.04t}
\]

Account 2: \( P = 3000, \quad r = .05, \quad n = 12, \quad \) compounded monthly
\[
A = P\left(1 + \frac{r}{n}\right)^{nt} = 3000\left(1 + \frac{.05}{12}\right)^{12t}
\]

Equate the right hand sides:
\[
2000\left(e^{.04}\right)^{t} = 3000\left[\left(1 + \frac{.05}{12}\right)^{12}\right]^{t}
\]
\[
2000\left(\frac{1.04081}{1.05116}\right)^{t} = 3000\left(\frac{1.05116}{1.05116}\right)^{t}
\]

Account on the left (Account 1) starts lower and has a lower APY than the account on the right (Account 2), so the accounts will never be equal in the future. We continue to see how the algebra plays out.
\[
\frac{1.04081^{t}}{1.05116^{t}} = \frac{3000}{2000}
\]
\[
\left(\frac{1.04081}{1.05116}\right)^{t} = \frac{3}{2}
\]
\[
\ln\left(\frac{1.04081}{1.05116}\right)^{t} = \ln\left(\frac{3}{2}\right)
\]
\[
t\ln\left(\frac{1.04081}{1.05116}\right) = \ln\left(\frac{3}{2}\right)
\]
\[
t = \frac{\ln\left(\frac{3}{2}\right)}{\ln\left(\frac{1.04081}{1.05116}\right)} = -40.98
\]

This shows that the accounts could only have been equal in the past.
**Problem (Page 404 #30).**

The lender receives $593.10 after 14 days.

\[
P = 500, \quad A = 593.10, \quad t = \frac{14}{365}
\]

\[
A = P \left(1 + \frac{r}{n}\right)^{nt} = P \left[\left(1 + \frac{r}{n}\right)^n\right]^t = Pb^t
\]

\[
593.10 = 500b^{14/365}
\]

\[
1.1862 = b^{14/365}
\]

\[
1.1862^{365/14} = (b^{14/365})^{365/14} = b
\]

\[
b = 85.78
\]

\[
APY = b - 1 = 84.78 = 8478\%
\]

**Problem (Page 404 #34).**

\[
P = 81.9, \quad A = 234.7, \quad t = 20
\]

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

(a)

\[
234.7 = 81.9(1 + r)^{20}
\]

\[
\frac{234.7}{81.9} = (1 + r)^{20}
\]

\[
1 + r = \left(\frac{234.7}{81.9}\right)^{1/20}
\]

\[
APY = r = \left(\frac{234.7}{81.9}\right)^{1/20} - 1 = .05405 = 5.405\%
\]

Tomato prices were increasing by 5.405% per year.

(b)

\[
A = P(1 + r)^t = .86(1.05405)^{20} = 2.46
\]

Tomatoes would be $2.46 per pound in 2010.
Problem (Page 405 #42).

\[ P = 25, \quad A = 125.62, \quad n = 1 \quad t = 23 + \frac{1}{12} = \frac{277}{12} \]

\[ 125.62 = 25(1 + r)^{277/12} \]

\[ \frac{125.62}{25} = (1 + r)^{277/12} \]

\[ 1 + r = \left( \frac{125.62}{25} \right)^{12/277} = 1.07244 \]

\[ APY = r = .07244 = 7.244\% \]

3. Future Value of an Increasing Annuity

An annuity is a financial instrument that requires a series of payments of set size and frequency.

For an ordinary annuity, payments are made at the end of each time period.

For an annuity due, payments are made at the beginning of each time period. These have slightly different formulas, but we will not study these in this course.

An increasing annuity is one where the balance increases with time.
EXAMPLE. If you deposit $100 per quarter at 8% compounded quarterly for one year, how much money is in the account at the end of the year?

For instance, how much money does payment 2 result in at the end of the year?

\[
\begin{array}{cccc|c}
0 & 1 & 2 & 3 & 4 \text{ quarters} \\
100 & & & & A
\end{array}
\]

We note that this payment receives two quarters of interest.

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt} = 100 \left( 1 + \frac{.08}{4} \right)^{4(\frac{1}{2})} = 100(1 + .02)^2
\]

Thus \( A = 100(1.02)^2 \)

Similarly,

\[
\begin{array}{cccc|c}
0 & 1 & 2 & 3 & 4 \text{ quarters} \\
100 & 100 & 100 & 100 & 100
\end{array}
\]

The amount we are looking for is the sum of the numbers in the right hand column. But we are going to add them a special way that helps us to generate a formula.

\[
S = 100 + 100(1.02) + 100(1.02)^2 + 100(1.02)^3
\]

\[
1.02S = 100(1.02) + 100(1.02)^2 + 100(1.02)^3 + 100(1.02)^4
\]

Subtract the top equation from the bottom one. Notice the cancelling.

\[
1.02S - S = 100(1.02)^4 - 100
\]

\[
.02S = 100(1.02^4 - 1)
\]

\[
S = 100 \frac{1.02^4 - 1}{.02} = 412.16
\]

Based on this work, we make the following definitions.
**Definition.** Future Value of an Increasing Annuity.

1. **With a Zero Present Value.**
   The future value $FV$ of an increasing annuity with an initial balance of 0 dollars is given by
   
   $$FV = PMT \left(\frac{(1+i)^m - 1}{i}\right)$$

   where $i = \frac{r}{n}$ is the periodic interest rate, $m = nt$ is the number of payments, and $PMT$ is the payment amount.

2. **With a Nonzero Present Value.**
   The future value $FV$ of an increasing annuity with a present value $PV$ is given by
   
   $$FV = PV(1+i)^m + PMT \left(\frac{(1+i)^m - 1}{i}\right).$$

**Example.** We want $1,000,000 after 40 years of monthly contributions at 12% compounded monthly. What is the monthly payment?

$$FV = 1,000,000, \quad i = \frac{12}{12} = .01, \quad m = 12(40) = 480$$

$$PMT = \frac{FV(i)}{(1+i)^m - 1} = \frac{[1,000,000(.01)]}{[(1+.01)^{480} - 1]} = \$85.00$$

**Problem (Page 415 #12).**

$$i = \frac{.0811}{12}, \quad PV = 3200, \quad m = 12(25) = 300, \quad PMT = 125$$

$$FV = PV(1+i)^m + PMT \left(\frac{(1+i)^m - 1}{i}\right)$$

$$FV = 3200 \left(1 + \frac{.0811}{12}\right)^{300} + 125 \left[\left(1 + \frac{.0811}{12}\right)^{300} - 1\right] = \$145,164.60$$

**Note.** Using $i = .00676$ gives an answer of $\$145,216.04.$
Problem (Page 415 #10 adjusted).

Suppose also that the account has a $5000 present value.

\[
PV = 5000, \quad i = \frac{.08}{12} \approx .00676, \quad FV = 250,000, \quad PMT = 300
\]

\[
FV = PV(1 + i)^m + PMT \frac{(1 + i)^m - 1}{i}
\]

\[
250,000 = 5000(1 + .00676)^m + 300 \left[ \frac{(1 + .00676)^m - 1}{.00676} \right]
\]

\[
250,000 = 5000(1.00676)^m + 44776.1194 (1.00676^m - 1)
\]

\[
250,000 = 5000(1.00676)^m + 44776.1194 (1.00676^m) - 44776.1194
\]

\[
294776.1194 = 5000(1.00676)^m + 44776.1194 (1.00676^m)
\]

\[
294776.1194 = (5000 + 44776.1194)(1.00676)^m
\]

\[
294776.1194 = 49776.1194(1.00676)^m
\]

\[
5.922 = 1.00676^m
\]

\[
\ln 5.922 = \ln 1.00676^m
\]

\[
\ln 5.922 = m \ln 1.0067
\]

\[
m = \frac{\ln 5.922}{\ln 1.0067} = 266.36 \text{ or } 267 \text{ months}
\]

Note. Without rounding off \(i\), we get \(m = 283\) months.

A sinking fund is an account established for the purpose of accumulating money to pay off future debts or obligations. It is always an increasing annuity.
The time value of money (TVM) refers to the fact that a given amount now is worth more than the same amount in the future since money now can gain interest in the interim.

**Problem** (Page 416 #22).

We use the TVM Solver as described on page 411. Note that for increasing annuities, \( PV \) and \( PMT \) are entered as negatives.

\[
N = \text{number of payments} = 21 \times 12 \\
PV = \text{negative of } PV = -10,185.22 \\
PMT = \text{negative of } PMT = -650 \\
FV = \text{future value} = 2,900,000 \\
P/Y = \text{payments per year} = 12 \\
C/Y = \text{compound periods per year} = 12 \\
PMT = \text{beginning or end of period for interest calculation} = END \\
I\% = \text{annual interest rate} \\
I = 19.55\%
\]

**Problem** (Page 417 #28).

Find \( N, \ I\% = 4.69, \ PV = -24020, \ PMT = -1000, \ FV = .20 \times 525000 - 20000, \ P/Y = 12, \ C/Y = 365, \ END \)

It will take 51 months (round up).

4. **Present Value of an Decreasing Annuity**

Amortization is the process of paying off a loan, usually with equal periodic payments.

The present value of an annuity is the amount of money that would need to be invested today in order to provide the desired numbers of equal periodic payments.
**Example.** In order to withdraw $100 for 4 quarters at 8% compounded quarterly, how much money must one start with?

For instance, how much money is needed at the beginning for payment 3 — a compound interest problem?

\[
P \begin{array}{cccc}
0 & 1 & 2 & 3 \\
100 & 100 & 100 & 100 \\
\end{array}
\]

\[
A = P \left(1 + \frac{r}{n}\right)^{nt} \implies 100 = P \left(1 + \frac{.08}{4}\right)^{4(3)} \implies P = \frac{100}{(1 + .02)^3} \implies P = 100(1.02)^{-3}
\]

Similarly,

\[
P = 100(1.02)^{-1} + 100(1.02)^{-2} + 100(1.02)^{-3} + 100(1.02)^{-4}
\]

\[
1.02P = 100 + 100(1.02)^{-1} + 100(1.02)^{-2} + 100(1.02)^{-3}
\]

Subtract the top equation from the bottom one. Notice the cancelling.

\[
1.02P - P = 100 - 100(1.02)^{-4}
\]

\[
.02P = 100(1 - 1.02^{-4})
\]

\[
P = 100 \frac{1 - 1.02^{-4}}{.02} = 380.77
\]
Based on this work, we make the following definitions.

**Definition.** Present Value of a Decreasing Annuity.

(1) With a Future Value of Zero.

The present value \( PV \) of an decreasing annuity with a future value of 0 dollars is given by

\[
PV = PMT \frac{1 - (1 + i)^{-m}}{i}
\]

where \( i = \frac{r}{n} \) is the periodic interest rate, \( m = nt \) is the number of payments, and \( PMT \) is the payment amount.

(2) With a Nonzero Future Value.

The present value \( PV \) of an decreasing annuity with a future value \( FV \) is given by

\[
PV = FV(1 + i)^{-m} + PMT \frac{1 - (1 + i)^{-m}}{i}
\]

(3) The payment \( PMT \) required to amortize (pay off) a debt of \( PV \) dollars is given by

\[
PMT = \frac{i(PV)}{1 - (1 + i)^{-m}}
\]

**Problem (Page 433 #6).**

\[
PV = 49330 - 18500 = 30830, \quad i = \frac{.0699}{12} = .005825, \quad m = 48
\]

\[
PMT = \frac{.005825(30830)}{[1 - (1.005825)^{-48}]} = 738.12
\]

See the associated amortization schedule:
4. PRESENT VALUE OF AN DECREASING ANNUITY

Home Amortization at 9.6%

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Principal (Present Value)</th>
<th>Interest (0.005825 x Principal)</th>
<th>New Balance (Principal + Payment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$30,830.00</td>
<td>$179.58</td>
<td>$30,271.46</td>
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<td>$4,339.79</td>
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<td>$1,463.42</td>
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<tr>
<td>48</td>
<td>$733.82</td>
<td>$4.27</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

$4,599.73 $35,429.73
**Problem (Page 433 #14).**

\[
i = \frac{.0685}{12}, \quad PV = .80(220,000), \quad m = 360
\]

\[
PMT = \frac{\left(\frac{.0685}{12}\right)(176,000)}{1 - \left(1 + \frac{.0685}{12}\right)^{-360}} = 1153.26
\]

**Problem (Page 435 #30).**

\[
PV = 300,000, \quad FV = 60,000, \quad i = \frac{.06}{12} = .005, \quad PMT = 4000
\]

\[
PV = FV(1 + i)^{-m} + PMT \frac{1 - (1 + i)^{-m}}{i}
\]

\[
300000 = 60000(1.005)^{-m} + 4000 \left[\frac{1 - (1.005)^{-m}}{.005}\right]
\]

\[
300000 = 60000(1.005)^{-m} + 800000 \left[1 - (1.005)^{-m}\right]
\]

\[
300000 = 60000(1.005)^{-m} + 800000 - 800000(1.005)^{-m}
\]

\[
-500000 = 60000(1.005)^{-m} - 800000(1.005)^{-m}
\]

\[
-500000 = -740000(1.005)^{-m}
\]

\[
\frac{50}{74} = 1.005^{-m}
\]

\[
\ln\left(\frac{50}{74}\right) = \ln 1.005^{-m}
\]

\[
\ln\left(\frac{50}{74}\right) = -m \ln 1.005
\]

\[
m = -\frac{\ln\left(\frac{50}{74}\right)}{\ln 1.005} = 78.6 \text{ or } 79 \text{ payments}
\]
Estimate of the Future Balance of a Credit Card.
The balance on a credit card, $B_n$, after $n$ minimum payments have been made may be estimated by

$$B_n = B(1 + i - r)^n$$

where $B$ is the initial balance, $i$ is the monthly periodic interest rate, and $r$ is the percentage of the balance that is the minimum amount required to be paid.

**Problem (Page 435 #32).**

$$B = 1890.25, \quad i = \frac{.219}{12}, \quad r = .0225, \quad n = 24$$

$$B_{24} = 1890.25 \left(1 + \frac{.219}{12} - .0225\right)^{24} = 1706.58$$

What about after 10 years?

$$B_{120} = 1890.25 \left(1 + \frac{.219}{12} - .0225\right)^{120} = 1133.85$$

An underestimate of how much we have paid:

monthly payment $= .0225(1133.85) = 25.51$

120 monthly payments $= 120(25.51) = 3061.20$

**Problem (Page 435 #36).**

There are two parts to this problem:

<table>
<thead>
<tr>
<th>saving</th>
<th>spending</th>
</tr>
</thead>
<tbody>
<tr>
<td>age 21</td>
<td>65</td>
</tr>
<tr>
<td>year 0</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>25</td>
</tr>
</tbody>
</table>

(1) Find out how much money you need for the withdrawals.
(2) Find the monthly payment to get that amount.
(1) Decreasing annuity:

\[ PMT = 3500, \quad i = \frac{0.08}{12}, \quad m = 25(12) = 300 \]

\[ PV = PMT \frac{1 - (1 + i)^{-m}}{i} = 3500 \left[ 1 - \left(1 + \frac{0.08}{12}\right)^{-300}\right] = $453,475.83 \]

(2) Increasing annuity:

\[ FV = 453,475.83, \quad i = \frac{0.08}{12}, \quad m = 44(12) = 528 \]

\[ FV = PMT \frac{(1 + i)^m - 1}{i} \]

\[ 453,475.83 = PMT \left[ \left(1 + \frac{0.08}{12}\right)^{528} - 1 \right] \]

\[ 453,475.83 = PMT (4858.81) \]

\[ PMT = $93.34 \]

Note.

\[ I = 300(3500) - 528(93.94) = 1,050,000 - 49,283.52 = $1,000,716.48 \]
What about?

<table>
<thead>
<tr>
<th>age</th>
<th>saving</th>
<th>letting it sit</th>
<th>spending</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21</td>
<td>30</td>
<td>65</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>35</td>
<td>25</td>
</tr>
</tbody>
</table>

(1) Same as before.

(2) Money just gains compound interest:

\[ A = 453,475.83, \quad i = \frac{.08}{12}, \quad m = 35(12) = 420 \]

\[ A = P(1 + i)^m \]

\[ 453,475.83 = P \left(1 + \frac{.08}{12}\right)^{420} \]

\[ 453,475.83 = P(16.29) \]

\[ P = \$27,837.69 \]

(3) Increasing annuity:

\[ FV = 27,837.69, \quad i = \frac{.08}{12}, \quad m = 9(12) = 108 \]

\[ FV = PMT \frac{(1 + i)^m - 1}{i} \]

\[ 27,837.69 = PMT \frac{\left[(1 + \frac{.08}{12})^{108} - 1\right]}{\left(\frac{.08}{12}\right)} \]

\[ 27,837.69 = PMT (157.42) \]

\[ PMT = \$176.84 \]

**Note.**

\[ I = 1,050,000 - 108(176.84) = 1,050,000 - 19,098.72 = \$1,030,901.28 \]