Lagrange’s Theorem and Consequences

**Theorem (7.1 — Lagrange’s Theorem: \(|H|\) Divides \(|G|\)).** If \(G\) is a finite group and \(H \leq G\), then \(|H||G|\). Moreover, the number of distinct left (right) cosets of \(H\) in \(G\) is \(\frac{|G|}{|H|}\).

**Proof.**

Let \(a_1H, a_2H, \ldots, a_rH\) denote the distinct left cosets of \(H\) in \(G\). Then, for all \(a \in G\), \(aH = a_iH\) for some \(i = 1, 2, \ldots, r\). By (1) of the Lemma, \(a \in aH\). Thus

\[
G = a_1H \cup a_2H \cup \cdots \cup a_rH.
\]

By (5) of the Lemma, this union is disjoint, so

\[
|G| = |a_1H| + |a_2H| + \cdots + |a_rH| = r|H|
\]

since \(|a_iH| = |aH|\) for \(i = 1, 2, \ldots r\).

**Definition.** The index of a subgroup \(H\) in \(G\) is the number of distinct left cosets of \(H\) in \(G\), and is denoted by \(|G : H|\).

**Corollary (1 — \(|G : H| = \frac{|G|}{|H|}\)).** If \(G\) is a finite group and \(H \leq G\), then \(|G : H| = \frac{|G|}{|H|}\).

**Proof.** Immediate consequence of Lagrange’s Theorem.

**Corollary (2 — \(|a|\) Divides \(|G|\)).** In a finite group, the order of each element divides the order of the group.

**Proof.** For \(a \in G\), \(|a| = |\langle a \rangle|\), so \(|a||G|\).