

**Example.**

(1) $\mathbb{Z}$ is a commutative ring with *unity* 1. 1 and $-1$ are the only units.

(2) $\mathbb{Z}_n$ with addition and multiplication modulo $n$ is a commutative ring with identity. The set of units is $U(n)$.

(3) The set $\mathbb{Z}[x]$ of all polynomials in $x$ with integer coefficients under ordinary addition and multiplication of polynomials is a commutative with identity $f(x) = 1$.

(4) $M_2(\mathbb{Z})$, the set of $2 \times 2$ matrices with integer entries is a noncommutative ring with unity \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

(5) The set $3\mathbb{Z}$ of multiples of 3 under ordinary addition and multiplication is a commutative ring without identity.

(6) $F = \{ f : \mathbb{R} \to \mathbb{R} \}$ is a commutative ring with identity where

$$
(f + g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x).
$$

$f(x) = 0$ is the zero function and $g(x) = 1$ is the identity.

(7) Let $R_1, R_2, \ldots, R_n$ be rings. Then

$$
R_1 \oplus R_2 \oplus \cdots \oplus R_n = \{(a_1, a_2, \ldots, a_n) | a_i \in R_i\}
$$

is a ring with componentwise addition and multiplication, i.e.,

$$
(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)
$$

and

$$
(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n).
$$

This is the **direct sum** of $R_1, R_2, \ldots, R_n$. 