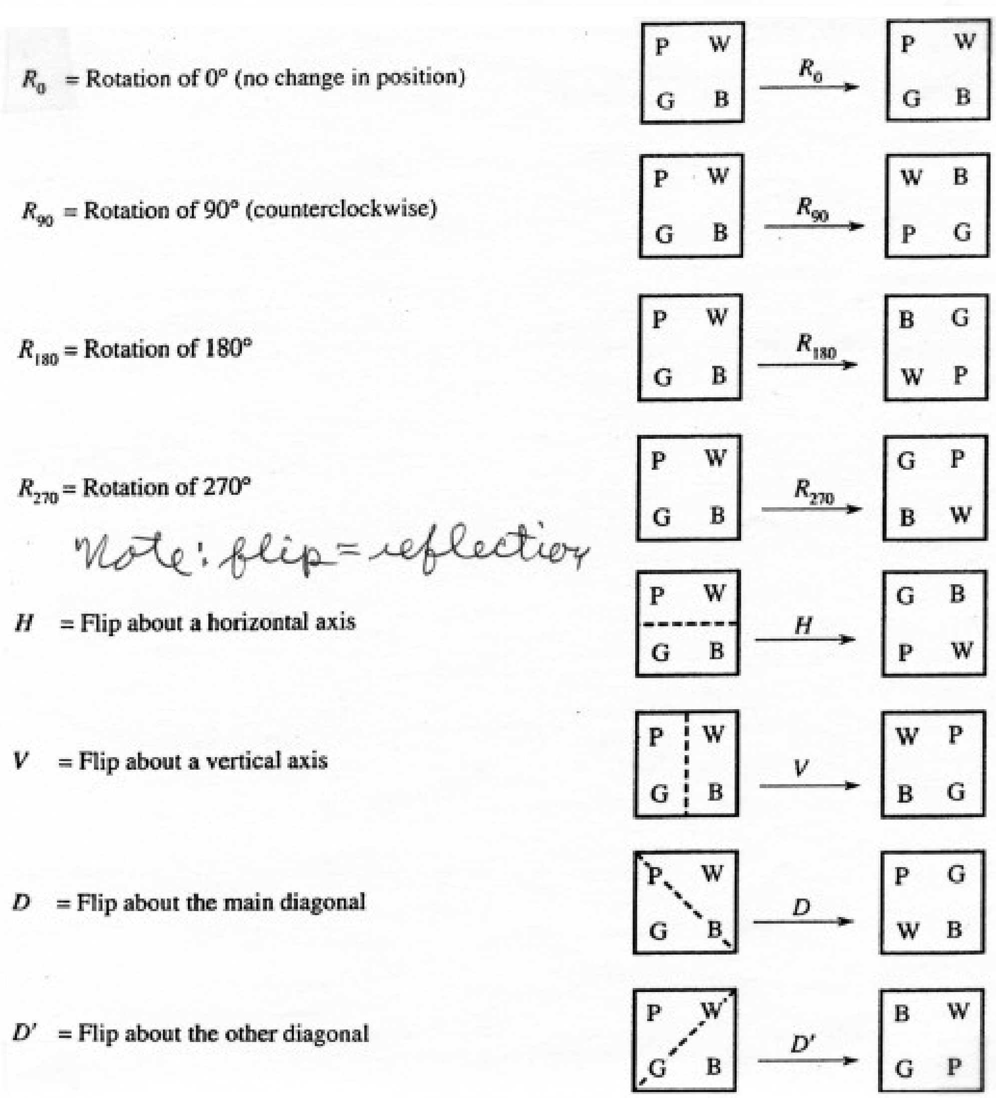
Introduction to Groups

Symmetries of a Square

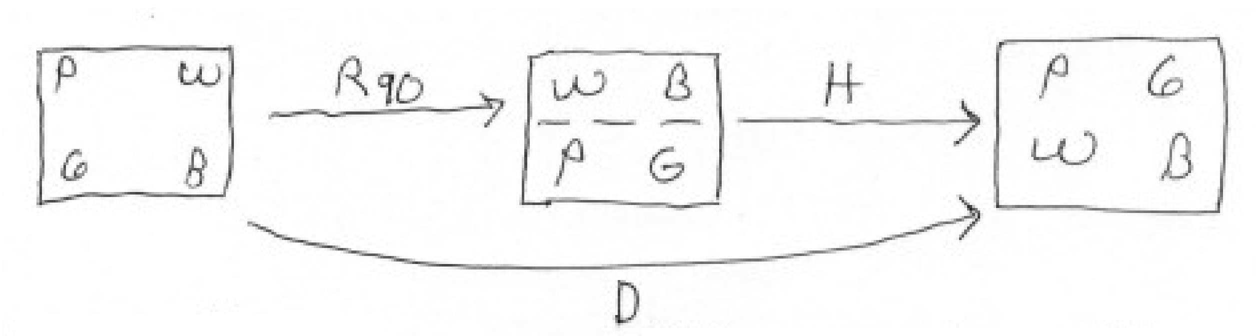
A plane symmetry of a square (or any plane figure F) is a function from the square to itself that preserves distances, i.e., the distance between the images of points P and Q equals the distance between P and Q.

The eight symmetries of a square:



These are all the symmetries of a square. Think of a square cut from a piece of glass with dots of different colors painted on top in the four corners. Then a symmetry takes a corner to one of four corners with dot up or down — 8 possibilities.

A succession of symmetries is a symmetry:



Since this is a composition of functions, we write $HR_{90} = D$, allowing us to view composition of functions as a type of multiplication. Since composition of functions is associative, we have an operation that is both closed and associative. R_0 is the identity: for any symmetry S, $R_0S = SR_0 = S$. Each of our symmetries S has an inverse S^{-1} such that $SS^{-1} = S^{-1}S = R_0$, our identity.

$$R_0R_0 = R_0, \quad R_{90}R_{270} = R_0, \quad R_{180}R_{180} = R_0, \quad R_{270}R_{90} = R_0$$
$$HH = R_0, \quad VV = R_0, \quad DD = R_0, \quad D'D' = R_0$$

We have a group.

This group is denoted D_4, and is called the dihedral group of order 8 (the number of elements in the group) or the group of symmetries of a square.

We view the Cayley table or operation table for D_4:

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

For $HR_{90} = D$ (circled), we find H along the left and R_{90} on top. The result of this operation (multiplication, composition) is D.

Notice $R_{90}H = D'$, so $HR_{90} \neq R_{90}H$ and our operation is not always commutative.

Also notice that our table is a Latin square — each element appears once in each row and each column.

Can you find a smaller group within this group — a subgroup?

$\{R_0, R_{90}, R_{180}, R_{270}\}$ — the cyclic rotation group of order 4 is commutative or Abelian.

Any others?

$\{R_0\}$

$\{R_0, R_{180}\}$

$\{R_0, H\}$ $\{R_0, V\}$ $\{R_0, D\}$ $\{R_0, D'\}$

$\{R_0, R_{180}, H, V\}$ $\{R_0, R_{180}, D, D'\}$

All of these are Abelian.

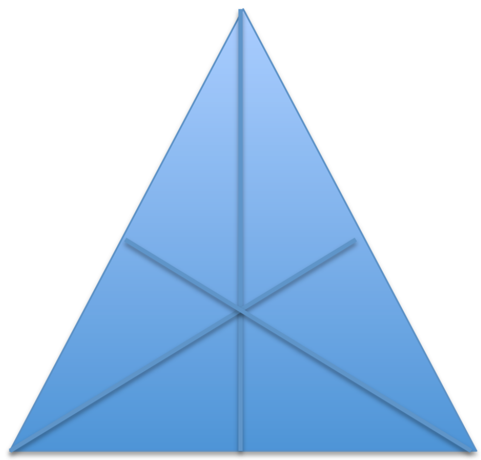
The entire group is also a subgroup of itself.

MAPLE. See dihedral8.mw or dihedral8.pdf.

D_4 has rotational and reflexive symmetry. So do all the dihedral groups of order $2n$, which are denoted by D_n, for $n \geq 3$, and are the symmetries of regular n-gons.

For n even, like for the square, axes of symmetry are lines joining midpoints of opposite sides or lines joining opposing vertices.

For n odd, axes of symmetry are lines joining a vertex to the midpoint of the opposite side.



A cyclic group of order n is denoted $\langle R_{360/n}\rangle$, and consists of

$$\left\{R_0, R_{\frac{360}{n}}, R_{\frac{2*360}{n}}, R_{\frac{3*360}{n}}, \ldots, R_{\frac{(n-1)*360}{n}}\right\}.$$

Thus

$$\langle R_{90}\rangle = \left\{R_0, R_{90}, R_{180}, R_{270}\right\}.$$

Check the logos with cyclic rotation symmetry groups at the top of page 37.

Problem (Page 38 # 6, 7). R_0 fixes all points (takes each point to itself). Any other rotation leaves only the center of the rotation fixed. A reflection leaves the axis of symmetry fixed.

(6) A reflection followed by a different reflection leaves a single point, the intersection of the two axes of symmetry, fixed, so is a rotation. A reflection followed by itself is R_0, also a rotation.

(7) A rotation followed by a rotation fixes a single point or is R_0, so in either case is a rotation.