CHAPTER 12

Functions of Several Variables and Partial Differentiation

1. Functions of Several Variables

**Definition.** A function of two variables is a rule that assigns a real number \( f(x, y) \) to each pair of real numbers \((x, y)\) in the domain of the function. We write

\[
f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}.
\]

**Example.**

\[
f(x, y) = e^x \cos y
\]

**Note.** This definition easily expands to 3 or more variables.

**Example.**

\[
f(x, y, z) = x^2 \cos y \sin z
\]

---

**Domains**

Unless specifically stated otherwise, the domain of a function of several variables is the set of all values of the variables for which the given expression is defined.

**Example.**

\[
f(x, y) = \frac{[\ln(x - 3)] \sqrt{y + 2}}{x^2 - 4}
\]

Cannot have \(x = \pm 2\).

Need \(x > 3\).

Need \(y \geq -2\).

\[
D_f = \left\{(x, y) \in \mathbb{R}^2 : x > 3 \text{ and } y \geq -2\right\}
\]
Graphing Functions $z = f(x, y)$ of Two Variables

**Maple.** See `func2var(12.1).mw` or `func2var(12.1).pdf`.

See Matching functions (matchfunctions.jpg).

**Contour Plots**

A level curve or contour of $f(x, y)$ is the 2-dimensional graph of the equation

$$f(x, y) = c.$$ 

These are obtained from a surface by slicing it with horizontal planes.

A contour plot of $f(x, y)$ is a graph of numerous level curves

$$f(x, y) = c,$$

for representative values of $c$.

A density plot shades each pixel according to the size of the function value at the point representing the pixel, with different colors and shades representing different function values.

**Maple.** See `contour(12.1).mw` or `contour(12.1).pdf`.

See Matching Functions and Contours (matchcontour.jpg).

See Dali’s Target (Dali.jpg).

Graphing Functions $w = f(x, y, z)$ of 3 Variables

Since we cannot see in 4 dimensions, we can only graph level surfaces of the form

$$w = f(x, y, z) = c.$$ 

**Maple.** See `func3var(12.1).mw` or `func3var(12.1).pdf`.

See Level Surfaces (Level Surfaces.jpg).
2. Limits and Continuity

Let
\[ d((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2} \]
and
\[ d((x, y, z), (a, b, c)) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}. \]

**Definition.** Let \( f \) be defined on the interior of a circle centered at \((a, b)\), except possibly at \((a, b)\) itself.

We say
\[ \lim_{(x, y) \to (a, b)} f(x, y) = L \]
if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ |f(x, y) - L| < \epsilon \]
whenever
\[ 0 < d((x, y), (a, b)) < \delta. \]

**Note.** This means \( f(x, y) \to L \) as \((x, y) \to (a, b)\) along any path toward \((a, b)\).
Below are some exotic paths for \((x, y) \to (0, 0)\)

\[ f(x, y) = \frac{3xy^2 + 6xy}{2x^2y + 4y} = \frac{3 \cdot 3 \cdot 4 + 6 \cdot 3 \cdot 2}{2 \cdot 9 \cdot 2 + 4 \cdot 2} = \frac{36 + 36}{36 + 8} = \frac{72}{44} = \frac{18}{11} \]

\textbf{Theorem.} If \(\lim_{(x,y)\to(a,b)} f(x, y) = L\) and \(\lim_{(x,y)\to(a,b)} g(x, y) = M\), then

\[(1) \lim_{(x,y)\to(a,b)} [f(x, y) \pm g(x, y)] = L \pm M \]

\[(2) \lim_{(x,y)\to(a,b)} [f(x, y)g(x, y)] = LM \]

\[(3) \lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \text{ provided } M \neq 0. \]

\textbf{Note.} If a polynomial in two variables is any sum of terms of the form \(cx^n y^m\), then the limit of the polynomial exists everywhere and is found simply by substitution.

\textbf{Example.}

\[ \lim_{(x,y)\to(3,2)} \frac{3xy^2 + 6xy}{2x^2y + 4y} = \frac{36 + 36}{36 + 8} = \frac{72}{44} = \frac{18}{11} \]

\textbf{Note.}

(1) If \(f(x, y) \to L_1\) along a path \(P_1\) toward \((a, b)\), and \(f(x, y) \to L_2 \neq L_1\) along a path \(P_2\) toward \((a, b)\), then

\[ \lim_{(x,y)\to(a,b)} f(x, y) \]

\(\text{does not exist (DNE).}\)
(2) The simplest paths to try when you suspect a limit does not exist are below. Either find one where a limit does not exist or two with different limits. If you expect the limit does exist, use one of these paths to find a value for the limit, then establish that limit by methods to be given below.

(a) vertical lines: \( x = a, y \to b \)

(b) horizontal lines: \( y = b, x \to a \)

(c) \( y = g(x) \) and \( x \to a \) while \( b = g(a) \)

(d) \( x = g(y) \) and \( y \to b \) where \( a = g(b) \)

**Example.** Find \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \).

For \( x = 0 \) (approaching the origin along the \( y \)-axis),
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y \to 0} \frac{-y^2}{y^2} = \lim_{y \to 0} (-1) = -1.
\]

But for \( y = 0 \) (approaching the origin along the \( x \)-axis),
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} (1) = 1.
\]

Since the limits along the two paths are different, \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does not exist (DNE).
Example. Find \( \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2} \).

For \( x = 0 \) (approaching the origin along the y-axis),
\[
\frac{2x^2y}{x^4 + y^2} = \frac{0}{y^2} = 0 \rightarrow 0 \text{ as } y \rightarrow 0.
\]

For \( y = 0 \) (approaching the origin along the x-axis),
\[
\frac{2x^2y}{x^4 + y^2} = \frac{0}{x^4} = 0 \rightarrow 0 \text{ as } x \rightarrow 0.
\]

But, along the parabola \( y = x^2 \),
\[
\frac{2x^2y}{x^4 + y^2} = \frac{2x^4}{x^4 + x^4} = \frac{2x^4}{2x^4} = 1 \rightarrow 1 \text{ as } (x, y) \rightarrow (0, 0).
\]

Since we now have different limits along two paths, \( \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2} \) does not exist (DNE).

Showing that a limit exists

Theorem (2.1). Suppose \( |f(x, y) - L| \leq g(x, y) \) for all \( (x, y) \) in the interior of some circle centered at \( (a, b) \), except possibly at \( (a, b) \).

If \( \lim_{(x,y)\to(a,b)} g(x, y) = 0 \), then \( \lim_{(x,y)\to(a,b)} f(x, y) = L \).
EXAMPLE. Find \( \lim_{(x,y) \to (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} \).

Along the path \( y = 0 \) (approaching the origin along the \( x \)-axis),
\[
\frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3x^2 + 5}{x^2 + 2} \to \frac{5}{2} \quad \text{as} \quad x \to 0,
\]
so \( \frac{5}{2} \) is the only possible limit. Then, using Theorem 2.1,
\[
\left| \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} - \frac{5}{2} \right| = \left| \frac{6x^2 - 2y^2 + 10 - 5x^2 - 5y^2 - 10}{2x^2 + 2y^2 + 4} \right| = \left| \frac{x^2 - 7y^2}{2x^2 + 2y^2 + 4} \right| \leq \frac{x^2 + 7y^2}{4} \to 0 \quad \text{as} \quad (x, y) \to (0, 0),
\]
so
\[
\lim_{(x,y) \to (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{5}{2}.
\]

EXAMPLE. Find \( \lim_{(x,y) \to (0,0)} \frac{3x^2|y|}{x^2 + y^2} \).

Along the path \( y = 0 \), \( \frac{3x^2|y|}{x^2 + y^2} = \frac{0}{x^2} = 0 \to 0 \quad \text{as} \quad x \to 0 \), so 0 is the only possible limit. Then, using Theorem 2.1,
\[
\left| \frac{3x^2|y|}{x^2 + y^2} - 0 \right| = \frac{3x^2|y|}{x^2 + y^2} = 3\frac{|y|}{x^2 + y^2} \leq 3|y| \to 0 \quad \text{as} \quad (x, y) \to (0, 0),
\]
so
\[
\lim_{(x,y) \to (0,0)} \frac{3x^2|y|}{x^2 + y^2} = 0.
\]
**Example.** Find \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \).

We use polar coordinates to find the indicated limit, if it exists. Note that \((x, y) \to (0, 0)\) is equivalent to \(r \to 0\). We have

\[
\lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \to 0} \frac{\sin(r^2)}{r^2} = (\text{by l’Hopitals’ rule})
\]

\[
\lim_{r \to 0} \frac{2r \cos(r^2)}{2r} = \lim_{r \to 0} \cos(r^2) = \cos 0 = 1.
\]

Thus \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \).

**Continuity**

**Definition.** Suppose \( f(x, y) \) is defined in the interior of a circle centered at \((a, b)\). We say \( f \) is **continuous** at \((a, b)\) if

\[
\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).
\]

If \( f(x, y) \) is not continuous at \((a, b)\), then we call \((a, b)\) a **discontinuity** of \( f \).

**Definition.**

(1) An **open disk** is the interior of a circle, i.e., all points inside but not on the circle.
(2) A **closed disk** is a circle and its interior.

(3) An **interior point** of a region \( R \subseteq \mathbb{R}^2 \) is the center of an open disk lying entirely within \( R \).

(4) A point is a **boundary point** of a region \( R \) if every open disk it is the center of contains points both in and out of \( R \).
(5) A **closed region** is a region that contains all of its boundary points.

(6) An **open region** contains none of its boundary points.

**Definition.** If the domain $D$ of a function contains any of its boundary points, and if $(a, b)$ is a boundary point of $D$, $f(x, y)$ is **continuous** at $(a, b)$ if

$$
\lim_{{(x,y) \to (a,b)}} f(x, y) = f(a, b).
$$

**Note.** If $f(x, y)$ and $g(x, y)$ are continuous at $(a, b)$, then $f + g$, $f - g$, and $f \cdot g$ are continuous at $(a, b)$; if, in addition, $g(a, b) \neq 0$, $\frac{f}{g}$ is continuous at $(a, b)$.

**Theorem (2.2).** Suppose $f(x, y)$ is continuous at $(a, b)$ and $g(x)$ is continuous at $f(a, b)$. Then

$$
h(x, y) = (g \circ f)(x, y) = g(f(x, y))
$$

is continuous at $(a, b)$.

**Example.** Where is $f(x, y) = \ln(x^2 + y^2)$ continuous?

$
\ln$

is continuous for all positive numbers.

Thus $f(x, y)$ is continuous except at $(x, y) = (0, 0)$.

**Example.** Where is $f(x, y) = \frac{1}{x^2 - y}$ continuous?

Since $f$ is a quotient of continuous functions, it is continuous except where $x^2 - y = 0$, i.e., except along the parabola $y = x^2$.

**Note.** All of the above transfers neatly to three variables.
**Definition.** Let $f$ be defined on the interior of a sphere centered at $(a, b, c)$, except possibly at $(a, b, c)$ itself. We say

$$\lim_{(x,y,z) \to (a,b,c)} f(x, y, z) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x, y, z) - L| < \epsilon$$

whenever

$$0 < d((x, y, z), (a, b, c)) < \delta.$$

**Example.** Find $\lim_{(x,y,z) \to (0,0,0)} \frac{(x + y + z)^2}{x^2 + y^2 + z^2}$.

For $x = y = 0$ (approaching the origin along the $z$-axis),

$$\lim_{(x,y,z) \to (0,0,0)} \frac{(x + y + z)^2}{x^2 + y^2 + z^2} = \lim_{z \to 0} \frac{z^2}{z^2} = \lim_{z \to 0} 1 = 1.$$

But for $x = y = z$ (These are symmetric equations for the line with parametric equations $x = t$, $y = t$, and $z = t$, or vector equation $\langle x, y, z \rangle = \langle 0, 0, 0 \rangle + t(1, 1, 1)$),

$$\lim_{(x,y,z) \to (0,0,0)} \frac{(x + y + z)^2}{x^2 + y^2 + z^2} = \lim_{x \to 0} \frac{(3x)^2}{3x^2} = \lim_{x \to 0} \frac{9x^2}{3x^2} = \lim_{x \to 0} (3) = 3.$$  

Since the limits along the two paths are different, $\lim_{(x,y,z) \to (0,0,0)} \frac{(x + y + z)^2}{x^2 + y^2 + z^2}$ does not exist (DNE).
2. LIMITS AND CONTINUITY

Example. Find \( \lim_{(x,y,z) \to (0,0,0)} \frac{x + y + z}{e^{x^2+y^2+z^2}}. \)

For \( x = y = 0 \) (approaching the origin along the \( z \)-axis),
\[
\frac{x + y + z}{e^{x^2+y^2+z^2}} = \frac{z}{e^{z^2}} \to 0 \text{ as } z \to 0,
\]
so 0 is the only possible limit.

\[
\left| \frac{x + y + z}{e^{x^2+y^2+z^2}} - 0 \right| \leq (\text{by } \Delta \text{ inequality}) \frac{|x|}{e^{x^2+y^2+z^2}} + \frac{|y|}{e^{x^2+y^2+z^2}} + \frac{|z|}{e^{x^2+y^2+z^2}} \leq \frac{|x|}{e^{x^2}} + \frac{|y|}{e^{y^2}} + \frac{|z|}{e^{z^2}} \to 0 + 0 + 0 = 0 \text{ as } (x, y, z) \to (0, 0, 0).
\]

Thus \( \lim_{(x,y,z) \to (0,0,0)} \frac{x + y + z}{e^{x^2+y^2+z^2}} = 0. \)

Definition. Suppose \( f(x, y, z) \) is defined in the interior of a sphere centered at \( (a, b, c) \). Then \( f \) is continuous at \( (a, b, c) \) if
\[
\lim_{(x,y,z) \to (a,b,c)} f(x, y, z) = f(a, b, c).
\]

If \( f(x, y, z) \) is not continuous at \( (a, b, c) \), then we call \( (a, b, c) \) a discontinuity of \( f \).

Example. \( f(x, y, z) = \frac{(x + y + z)^2}{x^2 + y^2 + z^2} \) is continuous everywhere except at \( (0, 0, 0) \) since it is the quotient of continuous functions. Similarly, \( f(x, y, z) = \frac{x + y + z}{e^{x^2+y^2+z^2}} \) is continuous everywhere since the denominator here is never 0.

Maple. See limit(12.2).mw or limit(12.2).pdf.
3. Partial Derivatives

Consider a function \( f(x, y) \) defined on a region \( R \subseteq \mathbb{R}^2 \). Let \((a, b)\) be an interior point of \( R \). The average rate of change as you move horizontally from \((a, b)\) to \((a + h, b)\) is

\[
\frac{f(a + h, b) - f(a, b)}{h}.
\]

The instantaneous rate of change in the \( x \)-direction at \((a, b)\) is

\[
\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},
\]

the partial derivative of \( f \) with respect to \( x \).
The average rate of change as you move vertically from \((a, b)\) to \((a, b + h)\) is
\[
\frac{f(a, b + h) - f(a, b)}{h}.
\]

The instantaneous rate of change in the \(y\)-direction at \((a, b)\) is
\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h},
\]
the partial derivative of \(f\) with respect to \(y\).
Example. Let \( f(x, y) = x^2y^2 \).

\[
\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{(x + h)^2y^2 - x^2y^2}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2y^2 + 2xhy^2 + h^2y^2 - x^2y^2}{h} = \lim_{h \to 0} \frac{2xhy^2 + h^2y^2}{h}
\]

\[
= \lim_{h \to 0} (2xy^2 + hy^2) = 2xy^2.
\]

Basically, hold \( y \) constant and take the derivative with respect to \( x \). We do similarly for \( \frac{\partial f}{\partial y}(x, y) \). Then, for example,

\[
\frac{\partial f}{\partial x}(3, 2) = 2 \cdot 3 \cdot 2^2 = 24.
\]

Now let’s estimate \( \frac{\partial f}{\partial x}(3, 2) \) using \( h = 0.1 \).

\[
\frac{\partial f}{\partial x}(3, 2) \approx \frac{f(3 + 0.1, 2) - f(3, 2)}{0.1} = \frac{38.44 - 36}{0.1} = 24.4.
\]

Notation.

\[
\frac{\partial f}{\partial x}(x, y) = \frac{\partial z}{\partial x}(x, y) = f_x(x, y) = \frac{\partial}{\partial x}[f(x, y)]
\]

\[
\frac{\partial f}{\partial y}(x, y) = \frac{\partial z}{\partial y}(x, y) = f_y(x, y) = \frac{\partial}{\partial y}[f(x, y)]
\]

Maple. See partderiv.mw or partderiv.pdf for graphic visualization and estimation from contour diagrams and tables.

Note. These definitions easily expand to 3 or more variables.
3. PARTIAL DERIVATIVES

Example.

\[ \frac{\partial f}{\partial x}(xe^{xy}) = \frac{\partial f}{\partial x}(xe^{(xy)^{1/2}}) = e^{(xy)^{1/2}} + xe^{(xy)^{1/2}}\left(\frac{1}{2}\right)(xy)^{-1/2}(y) = \left(\frac{1}{2}\right)e^{xy}(2 + \sqrt{xy}). \]

Note.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>1/2 z^{-1/2}</th>
<th>1/2</th>
<th>e^z</th>
<th>e^{z^{1/2}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>s</td>
<td>z</td>
<td>(xy)^{1/2}</td>
<td>e^s</td>
<td>e^{1/2}</td>
<td>e^{e^{1/2}}</td>
</tr>
</tbody>
</table>

Example. \( f(x, y) = x \ln(y \cos x) \). Find \( \frac{\partial f}{\partial x}(\frac{\pi}{3}, 1) \).

\[ \frac{\partial f}{\partial x}(x, y) = \ln(y \cos x) + x \frac{1}{y \cos x} (-y \sin x) = \ln(y \cos x) - x \tan x \]

Note.

<table>
<thead>
<tr>
<th>x</th>
<th>cos x</th>
<th>y</th>
<th>y cos x</th>
<th>ln(y cos x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-sin x</td>
<td>s</td>
<td>y</td>
<td>ys</td>
<td>ln(ys)</td>
</tr>
<tr>
<td>z</td>
<td>1/z</td>
<td>zn</td>
<td>ln z</td>
<td></td>
</tr>
</tbody>
</table>

Thus

\[ \frac{\partial f}{\partial x}\left(\frac{\pi}{3}, 1\right) = \ln \frac{1}{2} - \frac{\pi}{3} \sqrt{3} = -\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right). \]
2nd Order Partial Derivatives

\[ f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} \]

\[ f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \]

\[ f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \]

\[ f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} \]

**Note.** This method also extends to more variables and higher derivatives.

**Example.** \( f(x, y) = x^2 + x \ln y \)

\[ f_x(x, y) = 2x + \ln y \]

\[ f_y(x, y) = \frac{x}{y} \]

\[ f_{xx}(x, y) = 2 \]

\[ f_{xy}(x, y) = \frac{1}{y} \]

\[ f_{yx}(x, y) = \frac{1}{y} \]

\[ f_{yy}(x, y) = -\frac{x}{y^2} \]

**Note.** Notice that \( f_{xy} = f_{yx} \).

**Theorem (3.1).** If \( f_{xy} \) and \( f_{yx} \) are continuous at an interior point \((a, b)\) of their domain, then

\[ f_{xy}(a, b) = f_{yx}(a, b). \]
Visualizing 2nd Order Partial Derivatives

(1) \( f_{xx}(a, b) \)

\( f_x(a, b) < 0 \) since \( f \) is decreasing at \((a, b)\) for increasing \( x \).

\( f_{xx}(a, b) < 0 \) since the slice through \((a, b, 0) \parallel x\)-axis is concave down at \((a, b, f(a, b))\).

(2) \( f_{xy}(a, b) \)

\( f_x(a, b) < 0 \) since \( f \) is decreasing at \((a, b)\) for increasing \( x \).

\( f_{xy}(a, b) < 0 \) since the slope of successive tangent lines is decreasing for \( y \) increasing.
(3) $f_{yx}(a, b)$

$f_y(a, b) < 0$ since $f$ is decreasing at $(a, b)$ for increasing $y$. 

$f_{yx}(a, b) < 0$ since the slope of successive tangent lines is decreasing for $x$ increasing.

(4) $f_{yy}(a, b)$

$f_y(a, b) < 0$ since $f$ is decreasing at $(a, b)$ for increasing $y$. 

$f_{yy}(a, b) < 0$ since the slice through $(a, b, 0) \parallel y$-axis is concave down at $(a, b, f(a, b))$. 
Determining signs of 2nd order partial derivatives from a contour diagram.

Suppose \( z = f(x, y) \).

\( f_x(P) < 0 \) since \( f \) decreases as \( x \) increases (moving right from \( P \)).

\( f_y(P) > 0 \) since \( f \) increases as \( y \) increases (moving up from \( P \)).

\( f_{xx}(P) < 0 \) (green) since \( f \) decreases at an increasing rate as \( x \) increases. \( f_x \) decreases with increasingly closer contours.

**RECALL.** Closer contours mean a more rapid rate of change.

\( f_{yy}(P) < 0 \) (orange) since \( f \) increases at a decreasing rate as \( y \) increases. \( f_y \) decreases with increasingly spread contours.

\( f_{xy}(P) > 0 \) (blue) since \( f \) decreases at a decreasing rate as \( y \) increases. \( f_x \) increases with increasingly spread contours.

\( f_{yx}(P) > 0 \) (red) since \( f \) increases at a increasing rate as \( x \) increases. \( f_y \) increases with increasingly closer contours.

**Maple.** See 2ndorderpart(12.3).mw or 2ndorderpart(12.3).pdf.
Recall the linear approximation to a function $f(x)$ at $x = a$. This is the tangent line given by

$$y = f(a) + f'(a)(x - a).$$

For values of $x$ near $a$, we use this equation to approximate $f(x)$.

Similarly, we approximate values of $f(x, y)$ near the point $(a, b)$ by using the tangent plane to the surface $z = f(x, y)$ near the point $(a, b, f(a, b))$.

But what is the equation of this plane?

Now any vector tangent to the surface at this point lies in the tangent plane. Consider tangent vectors to the surface lying in planes parallel to the $x$-$z$ and $y$-$z$ planes. Their cross product would be a normal vector to the plane.
4. TANGENT PLANES AND LINEAR APPROXIMATIONS

\[ \mathbf{v}_1 \parallel \langle 1, 0, f_x(a, b) \rangle \quad \mathbf{v}_2 \parallel \langle 0, 1, f_y(a, b) \rangle \quad \mathbf{n} \parallel \langle f_x(a, b), f_y(a, b), -1 \rangle \]

We have

\[ \mathbf{v}_1 \parallel \langle 1, 0, f_x(a, b) \rangle \quad \text{and} \quad \mathbf{v}_2 \parallel \langle 0, 1, f_y(a, b) \rangle. \]

So we let

\[ \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(a, b) \\ 0 & 1 & f_y(a, b) \end{vmatrix} = \langle -f_x(a, b), -f_y(a, b), 1 \rangle. \]

**Theorem (4.1).** Suppose that \( f(x, y) \) has continuous first partial derivatives at \((a, b)\). A normal vector to the tangent plane to \( z = f(x, y) \) at \((a, b)\) is then

\[ \langle f_x(a, b), f_y(a, b), -1 \rangle. \]

Further, an equation of the tangent plane is given by

\[ z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

or

\[ z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \]

**Note.** The normal line to the surface at the point \((a, b, f(a, b))\) is

\[ x = a + f_x(a, b)t, \quad y = b + f_y(a, b)t, \quad z = f(a, b) - t \]

or

\[ \langle x, y, z \rangle = \langle a, b, f(a, b) \rangle + t \langle f_x(a, b), f_y(a, b), -1 \rangle. \]
Using the tangent plane, the linear approximation $L(x, y)$ of $f(x, y)$ at $(a, b)$ is

$$L(x, y) = z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

**NOTE.** Again, this all extends to 3 or more variables.

**EXAMPLE.** Suppose $f(x, y) = x^2y^3$. Find the tangent plane at $(1, 1, 1)$.

$$f_x(x, y) = 2xy^3 \implies f_x(1, 1) = 2.$$

$$f_y(x, y) = 3x^2y^2 \implies f_y(1, 1) = 3.$$

Since $f(1, 1) = 1$, the equation of the tangent plane at $(1, 1, 1)$ is

$$z = 1 + 2(x - 1) + 3(y - 1)$$

or

$$z = 2x + 3y - 4.$$

Then the linear approximation at $(1, 1)$ is

$$L(x, y) = 2x + 3y - 4.$$

Now $f(1.1, 1.1) = 1.61051$, and

$$L(1.1, 1.1) = 2.2 + 3.3 - 4 = 1.5$$

is an approximation with error $= 0.11051$. 

**Increments and Differentials**

Recall that the increment $\Delta y$ of $f(x)$ at $x = a$ is

$$\Delta y = f(a + \Delta x) - f(a),$$

and for $\Delta x$ “small,”

$$\Delta y \approx dy = f'(a)\Delta x.$$

For $z = f(x, y)$, we define the increment of $f$ at $(a, b)$ to be

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$
\[ \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = \]
\[ [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)] = \]
\[ f_x(u, b + \Delta y)[(a + \Delta x) - a] + f_y(a, v)[(b + \Delta y) - b] \text{ (by MVT)} = \]
\[ f_x(u, b + \Delta y)\Delta x + f_y(a, v)\Delta y = \]
\[ \left\{ f_x(a, b) + \underbrace{[f_x(u, b + \Delta y) - f_x(a, b)]}_{\epsilon_1} \right\} \Delta x + \]
\[ \left\{ f_y(a, b) + \underbrace{[f_y(a, v) - f_y(a, b)]}_{\epsilon_2} \right\} \Delta y = \]
\[ \underbrace{f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y}_{\text{error}}. \]

Since \( f_x \) and \( f_y \) are continuous,
\[ \epsilon_1 \to 0 \text{ and } \epsilon_2 \to 0 \text{ as } (\Delta x, \Delta y) \to (0, 0). \]
Theorem (4.2). Suppose $z = f(x, y)$ is defined on a rectangular region

$$R = \{(x, y)|x_0 < x < x_1, y_0 < y < y_1\}$$

and $f_x$ and $f_y$ are defined on $R$ and are continuous at $(a, b) \in R$. Then for

$$(a + \Delta x, b + \Delta y) \in R,$$

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where $\epsilon_1$ and $\epsilon_2$ are functions of $\Delta x$ and $\Delta y$ that tend to 0 as $(\Delta x, \Delta y) \to (0, 0)$.

Example. $f(x, y) = x^3 - 3xy$.

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= (x + \Delta x)^3 - 3(x + \Delta x)(y + \Delta y) - (x^3 - 3xy)$$

$$= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 3xy - 3x\Delta y$$

$$- 3y\Delta x - 3\Delta x\Delta y - x^3 + 3xy$$

$$= \underbrace{(3x^2 - 3y)}_{f_x}\Delta x + \underbrace{(-3x)}_{f_y}\Delta y + \underbrace{(3x\Delta x + (\Delta x)^2)}_{\epsilon_1}\Delta x + \underbrace{(-3\Delta x)}_{\epsilon_2}\Delta y$$

Letting $dx = \Delta x$ and $dy = \Delta y$, the differential of $z$ is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$  

This is called the total differential and we use it to approximate $\Delta z$, i.e.,

$$\Delta z \approx dz$$

when $dx = \Delta x$ and $dy = \Delta y$ are small.

Definition. Let $z = f(x, y)$. $f$ is differentiable at $(a, b)$ if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where both $\epsilon_1$ and $\epsilon_2$ are functions of $\Delta x$ and $\Delta y$ and

$$\epsilon_1 \to 0 \text{ and } \epsilon_2 \to 0 \text{ as } (\Delta x, \Delta y) \to (0, 0).$$

We say $f$ is differentiable on a region $R \subseteq \mathbb{R}^2$ if $f$ is differentiable at every point of $R$. 
Problem (Page 861 #40).

<table>
<thead>
<tr>
<th>Speed</th>
<th>Temp</th>
<th>30</th>
<th>20</th>
<th>10</th>
<th>0</th>
<th>−10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td>−10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>27</td>
<td>16</td>
<td>6</td>
<td>−5</td>
<td>−15</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>16</td>
<td>4</td>
<td>−9</td>
<td>−24</td>
<td>−33</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>−5</td>
<td>−18</td>
<td>−32</td>
<td>−45</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>−10</td>
<td>−25</td>
<td>−39</td>
<td>−53</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0</td>
<td>−15</td>
<td>−29</td>
<td>−44</td>
<td>−59</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>−2</td>
<td>−18</td>
<td>−33</td>
<td>−48</td>
<td>−63</td>
<td></td>
</tr>
</tbody>
</table>

The table here gives wind chill (how cold it feels outside) as a function of temperature (degrees Fahrenheit) and wind speed (mph). We can think of this as a function \( w(t, s) \). Using the table above, estimate the linear approximation of wind chill at \((10, 15)\) and use it to estimate the wind chill at \((12, 13)\).

(1) We estimate the linear approximation of wind chill at \((10, 15)\):

\[
w_t(10, 15) \approx \frac{-5 - (-32)}{20 - 0} = \frac{27}{20} = 1.35
\]

\[
w_s(10, 15) \approx \frac{-25 - (-9)}{20 - 10} = \frac{-16}{10} = -1.6
\]

Then

\[
w(t, s) \approx L(t, s) = -18 + 1.35(t - 10) - 1.6s - 15,
\]

so

\[
L(12, 13) = -18 + 1.35(2) - 1.6(-2)
\]

\[
= -18 + 5.9 = -12.1
\]

There is a rise of approximately 5.9 degrees in wind chill.
EXAMPLE. Consider the solid obtained when rotating the following region about the $x$-axis. Note that the region is composed of a right triangle and a quarter-circle.

(1) Compute the volume $V(x, y)$ of this solid.

$$V = \frac{1}{3} \pi r^2 h + \frac{1}{2} \left( \frac{4}{3} \pi r^3 \right) = \frac{1}{3} \pi (2^2)(3) + \frac{1}{2} \frac{14}{23} \pi (2^3) = \frac{28}{3} \pi \approx 29.3215$$

(2) Find the volume $V(x, y)$ of a similar solid created by rotating a region with horizontal dimension $x$ and vertical dimension $y$ instead of 3 and 2.

$$V = \frac{1}{3} \pi x^2 y + \frac{2}{3} \pi x^3$$

(3) Oh yeah — we forgot to tell you in (1) that the “2” and the “3” were really just rounded-off numbers. The actual quantities can be off by up to 0.5 in each direction. Use linear approximation to estimate the maximum possible error in your answer to (1).

$$|dx| \leq 0.5, \quad |dy| \leq 0.5$$

$$dV(x, y) = V_x(x, y)dx + V_y(x, y)dy$$

$$V_x(x, y) = \frac{2}{3} \pi xy + 2 \pi x^2 \implies V_x(3, 2) = \frac{2}{3} \pi (3)(2) + 2 \pi (3^2) = 22 \pi \approx 69.1150$$

$$V_y(x, y) = \frac{1}{3} \pi x^2 \implies V_y(3, 2) = \frac{1}{3} \pi (3^2) = 3 \pi \approx 9.4248$$

$$|dV(3, 2)| \leq 69.1150(0.5) + 9.4248(0.5) = 39.2699$$
Extending to 3 variables

**DEFINITION.** The linear approximation to \( f(x, y, z) \) at \((a, b, c)\) is
\[
L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).
\]
Also, if \( w = f(x, y, z) \),
\[
\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)
\approx f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.
\]

**MAPLE.** See linapprox(12.4).mw or linapprox(12.4).pdf.

## 9.4. Polar Coordinates

Cartesian and polar coordinates for a point \( P \) in the plane:

![Diagram of polar coordinates](image)

We have
\[
x = r \cos(\theta)
\]
\[
y = r \sin(\theta)
\]
\[
r = \pm \sqrt{x^2 + y^2} \text{ or } r^2 = x^2 + y^2
\]
\[
\tan(\theta) = \frac{y}{x} \text{ or } \theta = \arctan \left( \frac{y}{x} \right)
\]
Example.

(1) \( r = 2 \implies \pm \sqrt{x^2 + y^2} = 2 \implies x^2 + y^2 = 4. \)

This is a circle of radius 2.

(2) \( \theta = \frac{\pi}{3} \implies \tan \frac{\pi}{3} = \frac{y}{x} \implies \sqrt{3} = \frac{y}{x} \implies y = \sqrt{3}x. \)

This is a line through the origin with slope \( \sqrt{3} \).

Note.

\[
(2, \frac{\pi}{4}) = (-2, \frac{5\pi}{4}) = (-2, -\frac{3\pi}{4}),
\]
i.e., there are multiple representations for each point in polar coordinates.
EXAMPLE.

(1) Find the rectangular representation of \((r, \theta) = (-2, \frac{\pi}{3})\).

\[
x = -2 \cos \left(\frac{\pi}{3}\right) = -2 \cdot \frac{1}{2} = -1
\]
\[
y = -2 \sin \left(\frac{\pi}{3}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}
\]

Thus the Cartesian coordinates are \((-1, -\sqrt{3})\).

(2) Find all polar representations of \((2, -1)\).

\[
r^2 = x^2 + y^2 = 4 + 1 = 5 \implies r = \pm \sqrt{5}.
\]

We note the point is in the fourth quadrant.

\[
\tan \theta = \frac{y}{x} = -\frac{1}{2} \implies \theta = \arctan \left(-\frac{1}{2}\right) \text{ is in the fourth quadrant.}
\]

Thus the polar representations are

\[
(\sqrt{5}, \arctan \left(-\frac{1}{2}\right) + 2k\pi) \text{ or } (-\sqrt{5}, \arctan \left(-\frac{1}{2}\right) + (2k + 1)\pi), k \in \mathbb{Z}.
\]

(3) Convert \(r = 2 \sin \theta\) to a rectangular equation.

\[
r = 2 \sin \theta \implies r^2 = 2r \sin \theta \implies x^2 + y^2 = 2y.
\]

**Maple.** See polar(9.4).mw or polar(9.4).pdf.
4. TANGENT PLANES AND LINEAR APPROXIMATIONS

9.5. Calculus and Polar Coordinates

Suppose \( r = f(\theta) \). Then

\[
x = r \cos \theta = f(\theta) \cos(\theta) \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin(\theta)
\]

are parametric equations for the graph. Then, at \( \theta = a \),

\[
\left. \frac{dy}{dx} \right|_{\theta=a} = \frac{\frac{dy}{d\theta}(a)}{\frac{dx}{d\theta}(a)}.
\]

From the product rule,

\[
\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.
\]

Thus

\[
\left. \frac{dy}{dx} \right|_{\theta=a} = \frac{f'(a) \sin a + f(a) \cos a}{f'(a) \cos a - f(a) \sin a}.
\]

**Example.** Find the slope of the tangent line to \( r = \sin 4\theta \) at \( \theta = \frac{\pi}{16} \).

\[
f(\theta) = \sin 4\theta \implies f'(\theta) = 4 \cos 4\theta
\]

\[
\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{16}} = \frac{2\sqrt{2} \sin \frac{\pi}{16} + \sqrt{2} \cos \frac{\pi}{16}}{2\sqrt{2} \cos \frac{\pi}{16} - \sqrt{2} \sin \frac{\pi}{16}} = \frac{4 \sin \frac{\pi}{16} + \cos \frac{\pi}{16}}{4 \cos \frac{\pi}{16} - \sin \frac{\pi}{16}} \approx .4724
\]
5. The Chain Rule

Example.

(1) Sally runs twice as fast as Patty.
Patty runs 5 times as fast as Joey.
How much faster does Sally run than Joey?
Let \( s(t) \) = Sally’s position at time \( t \),
\( p(t) \) = Patty’s position at time \( t \),
\( j(t) \) = Joey’s position at time \( t \).

\[
\frac{ds}{dj} = \frac{ds}{dp} \cdot \frac{dp}{dj} = 2 \cdot 5 = 10 \text{ is the \textit{chain rule}.}
\]

(2) \( y = e^{\sin(3x)} \). Find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dz} \cdot \frac{dz}{dx} = e^w \cdot \cos z \cdot 3 = 3e^{\sin(3x)} \cos(3x)
\]
PROBLEM. Find \( \frac{d}{dt}[f(x, y)] \) if \( x \) and \( y \) are functions of \( t \).

Let \( g(t) = f(x(t), y(t)) \). Then

\[
\frac{d}{dt}[f(x(t), y(t))] = g'(t) = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))}{\Delta t}
\]

\[
\begin{cases}
\Delta x = x(t + \Delta t) - x(t), & \Delta y = y(t + \Delta t) - y(t), \\
\Delta z = f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))
\end{cases}
\]

Recall \( \Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \)

where \( \epsilon_1, \epsilon_2 \to 0 \) as \( (\Delta x, \Delta y) \to (0, 0) \)

\[
= \lim_{\Delta t \to 0} \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]

\[
= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \lim_{\Delta t \to 0} \epsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \epsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}.
\]

Now \( \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \).

Similarly, \( \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} \).

\( \lim_{\Delta t \to 0} \Delta x = \lim_{\Delta t \to 0} [x(t + \Delta t) - x(t)] = 0 \) since \( x(t) \) is continuous. Similarly,

\( \lim_{\Delta t \to 0} \Delta y = 0 \). Thus, since \( (\Delta x, \Delta y) \to (0, 0) \) as \( \Delta t \to 0 \), we have

\[
\lim_{\Delta t \to 0} \epsilon_1 = \lim_{\Delta t \to 0} \epsilon_2 = 0. \text{ Then}
\]

\[
\frac{d}{dt}[f(x(t), y(t))] = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} + 0 \cdot \frac{dy}{dt}
\]

\[
= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}
\]
Theorem (5.1 - Chain Rule). If \( z = f(x(t), y(t)) \), where \( x(t) \) and \( y(t) \) are differentiable and \( f(x, y) \) is a differentiable function of \( x \) and \( y \), then

\[
\frac{dz}{dt} = \frac{d}{dt} [f(x(t), y(t))] = \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}
\]

We can use a tree diagram to illustrate this.

\[
z = f(x, y)
\]

\[
\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}
\]

\[
x \quad y
\]

\[
\frac{dx}{dt} \quad \frac{dy}{dt}
\]

\[
t \quad t
\]

\( x \) and \( y \) are intermediate variables. Then \( \frac{dz}{dt} \) is the sum of all products of derivatives along each path to \( t \).
Example. \( z = f(x, y) = \frac{x}{y}, \) \( x = \sin t, \) \( y = \cos t. \) Find \( \frac{dz}{dt}. \)

First approach:

\[
z = \frac{x}{y} = \frac{\sin t}{\cos t} = \tan t \quad \Rightarrow \quad \frac{dz}{dt} = \sec^2 t.
\]

Second approach:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{1}{y} \cos t + \frac{x}{y^2} \sin t = 1 + \tan^2 t = \sec^2 t.
\]

We now extend the Chain Rule further.
Theorem (5.2 – Chain Rule). Suppose \( z = f(x, y) \) where \( f \) is a differentiable function of \( x \) and \( y \) and where \( x = x(s, t) \) and \( y = y(s, t) \) both have first order partial derivatives. Then

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}
\]

and

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.
\]
### Example

Suppose \( z = f(x, y) \) with \( x = r \cos \theta, y = r \sin \theta \). Find \( f_r, f_{rr} \).

We have

\[
\frac{\partial x}{\partial r} = \cos \theta \quad \text{and} \quad \frac{\partial y}{\partial r} = \sin \theta.
\]

\[
f_r = f_x \cdot \frac{\partial x}{\partial r} + f_y \cdot \frac{\partial y}{\partial r} = f_x \cos \theta + f_y \sin \theta
\]

\[
f_{rr} = \frac{\partial}{\partial r} (f_r) = \frac{\partial}{\partial r} (f_x \cos \theta + f_y \sin \theta)
\]

\[
= \frac{\partial}{\partial r} (f_x) \cos \theta + \frac{\partial}{\partial r} (f_y) \sin \theta
\]

\[
= \left( f_{xx} \frac{\partial x}{\partial r} + f_{xy} \frac{\partial y}{\partial r} \right) \cos \theta + \left( f_{yx} \frac{\partial x}{\partial r} + f_{yy} \frac{\partial y}{\partial r} \right) \sin \theta
\]

Use \( f_x \) for \( f \) in diagram above

Use \( f_y \) for \( f \) in diagram above

\[
= \left( f_{xx} \cos \theta + f_{xy} \sin \theta \right) \cos \theta + \left( f_{yx} \cos \theta + f_{yy} \sin \theta \right) \sin \theta
\]

\[
= f_{xx} \cos^2 \theta + 2 f_{xy} \sin \theta \cos \theta + f_{yy} \sin^2 \theta
\]
Problem (Page 870 #32).

The wave equation for the displacement $u(x,t)$ of a vibrating string of length $L$ is $a^2u_{xx} = u_{tt}, \ 0 < x < L$, for some constant $a^2$. Make the change of variables $X = \frac{x}{L}$ and $T = \frac{a}{L}t$ to simplify the equation.

$$u_x = u_X \frac{dX}{dx} = \frac{1}{L}u_X$$

$$u_t = u_T \frac{dT}{dt} = \frac{a}{L}u_T$$

$$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}\left(\frac{1}{L}u_X\right) = \frac{1}{L}\frac{\partial}{\partial x}(u_X) = \frac{1}{L}\left(u_{XX} \frac{dX}{dx}\right) = \frac{1}{L^2}u_{XX}$$

$$u_{tt} = \frac{\partial}{\partial t}(u_t) = \frac{\partial}{\partial t}\left(\frac{a}{L}u_T\right) = \frac{a}{L}\frac{\partial}{\partial t}(u_T) = \frac{a}{L}\left(u_{TT} \frac{dT}{dt}\right) = \frac{a^2}{L^2}u_{TT}$$

Then

$$a^2u_{xx} = u_{tt} \implies a^2\left(\frac{1}{L^2}u_{XX}\right) = \frac{a^2}{L^2}u_{TT} \implies u_{XX} = u_{TT}.$$ 

Assuming that $X$ and $T$ are dimensionless, find the dimensions of $a^2$.

If $x$ is in ft and $t$ is in sec, $a^2$ is in $\frac{\text{sec}^2}{\text{ft}^2}$. 

\[ \text{Therefore, the dimensions of } a^2 \text{ are } \frac{\text{sec}^2}{\text{ft}^2}. \]
**Implicit Differentiation**

Suppose $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$ where $f$ is differentiable. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Let

$$w = F(x, y, z) \implies \frac{\partial w}{\partial x} = F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x}.$$  

Now, since $w = 0$, $\frac{\partial w}{\partial x} = 0$. Also, $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$. Then

$$0 = F_x + F_z \frac{\partial z}{\partial x}.$$  

If $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}. $$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$  

**Theorem** (Implicit Function Theorem). If $F_x, F_y,$ and $F_z$ are continuous inside a sphere containing $(a, b, c)$ where $F(a, b, c) = 0$ and $F_z(a, b, c) \neq 0$, then $F(x, y, z) = 0$ implicitly defines $z$ as a function of $x$ and $y$ near $(a, b, c)$.
Problem (Page 870 #24). Suppose $3yz^2 - e^{4x} \cos 4z - 3y^2 = 4$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$F(x, y, z) = 3yz^2 - e^{4x} \cos 4z - 3y^2 - 4 = 0$$

$$F_x = -4e^{4x} \cos 4z, \quad F_y = 3z^2 - 6y, \quad F_z = 6yz + 4e^{4x} \sin 4z$$

$$\frac{\partial z}{\partial x} = \frac{F_x}{F_z} = \frac{4e^{4x} \cos 4z}{6yz + 4e^{4x} \sin 4z}$$

$$\frac{\partial z}{\partial y} = \frac{F_y}{F_z} = \frac{-3z^2 + 6y}{6yz + 4e^{4x} \sin 4z}$$

Maple. See chainrule(12.5).mw or chainrule(12.5).pdf.
6. The Gradient and Directional Derivatives

The partial derivatives $f_x$ and $f_y$ measure rates of change in directions parallel to the $x$- and $y$-axes. What about other directions?

Suppose we want the instantaneous rate of change of $f(x, y)$ at $P(a, b)$ in the direction given by the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$. Let $Q(x, y)$ be any point on the line through $P(a, b)$ in the direction of $\mathbf{u}$. Then

$$\frac{\mathbf{PQ}}{|| \mathbf{u} ||} \rightarrow \overrightarrow{PQ} = h \mathbf{u} \rightarrow \overrightarrow{PQ} = \langle x-a, y-b \rangle = h \mathbf{u} = h \langle u_1, u_2 \rangle = \langle hu_1, hu_2 \rangle.$$

Then

$$x - a = hu_1 \text{ and } y - b = hu_2 \rightarrow x = a + hu_1 \text{ and } y = b + hu_2.$$

Then the average rate of change from $P$ to $Q$ is

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

**Definition.** The directional derivative of $f(x, y)$ at the point $(a, b)$ and in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided the limit exists.

**Note.** If $\mathbf{u} = \langle 1, 0 \rangle$, $D_{\mathbf{u}}f(a, b) = f_x(a, b)$, and if $\mathbf{u} = \langle 0, 1 \rangle$, $D_{\mathbf{u}}f(a, b) = f_y(a, b)$. 
How to easily calculate directional derivatives.

Let
\[ g(h) = f(a + hu_1, b + hu_2) \implies g(0) = f(a, b). \]

Then
\[ D_uf(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0). \]

Now let
\[ x = a + hu_1 \text{ and } y = b + hu_2 \implies g(h) = f(x, y) \implies \]
\[ g'(h) = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = \frac{\partial f}{\partial x} \, u_1 + \frac{\partial f}{\partial y} \, u_2. \]

If \( h = 0 \),
\[ D_uf(a, b) = g'(0) = \frac{\partial f}{\partial x} \, (a, b)u_1 + \frac{\partial f}{\partial y} \, (a, b)u_2. \]

**THEOREM (6.1).** If \( f(x, y) \) is differentiable at \((a, b)\) and \( u = \langle u_1, u_2 \rangle \) is any unit vector,
\[ D_uf(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2. \]

**EXAMPLE.** \( f(x, y) = x^3 + y^2, \ u = \langle \frac{3}{5}, -\frac{4}{5} \rangle, (a, b) = (2, 1). \)
\( f_x = 3x^2, f_y = 2y, f_x(2, 1) = 12, f_y(2, 1) = 2. \)
\[ D_uf(2, 1) = 12 \left( \frac{3}{5} \right) + 2 \left( -\frac{4}{5} \right) = \frac{28}{5}. \]

**DEFINITION.** The **gradient** of \( f(x, y) \) is the vector-valued function
\[ \nabla f(x, y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \langle f_x, f_y \rangle \]
provided both partial derivatives exist.

**THEOREM (6.2).** If \( f \) is a differentiable function of \( x \) and \( y \) and \( u \) is any unit vector,
\[ D_uf(x, y) = \nabla f(x, y) \cdot u. \]
Properties of the gradient vector.

Suppose $\theta$ is the angle between $\nabla f(a, b)$ and $\mathbf{u}$.

Then
\[ D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = \|\nabla f(a, b)\| \cdot \|\mathbf{u}\| \cos \theta = \|\nabla f(a, b)\| \cdot \cos \theta. \]

Note that $D_{\mathbf{u}} f(a, b)$ is maximized when $\theta = 0$ or $\cos \theta = 1$. Also, $\theta = 0 \implies \nabla f(a, b)$ and $\mathbf{u}$ are in the same direction $\implies \mathbf{u} = \frac{\nabla f(a, b)}{\|\nabla f(a, b)\|}$.

Further, $D_{\mathbf{u}} f(a, b)$ is minimized when $\theta = \pi$ or $\cos \theta = -1$, i.e., $D_{\mathbf{u}} f(a, b)$ and $\mathbf{u}$ have opposite directions $\implies \mathbf{u} = - \frac{\nabla f(a, b)}{\|\nabla f(a, b)\|}$.

Finally, $\theta = \frac{\pi}{2} \implies \cos \theta = 0 \implies D_{\mathbf{u}} f(a, b) = 0 \implies \nabla f(a, b) \perp \mathbf{u} \implies \mathbf{u}$ is tangent to the level curve at $(a, b)$ since $f$ is constant on level curves and $\nabla f(a, b)$ is orthogonal to the level curve at $(a, b)$. 
Theorem (6.3). Suppose $f$ is a differentiable function of $x$ and $y$ at $(a, b)$. Then

1. the maximum rate of change of $f$ at $(a, b)$ is $\| \nabla f(a, b) \|$, occurring in the direction of the gradient;
2. the minimum rate of change of $f$ at $(a, b)$ is $-\| \nabla f(a, b) \|$, occurring in the direction opposite the gradient;
3. the rate of change of $f$ at $(a, b)$ is 0 in the directions orthogonal to $\nabla f(a, b)$;
4. the gradient $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = c$ at the point $(a, b)$ where $f(a, b) = c$.

Problem (Page 881 #58). At a certain point on a mountain, a surveyor sights due west and measures a $4^\circ$ rise, then sights due north and measures a $3^\circ$ rise. Find the direction of steepest ascent and compute the degree rise in that direction.

We let $z = f(x, y)$ and assume we are at the point $(x_0, y_0, z_0)$ on the mountain.

$$f_x(x_0, y_0) = -\tan 4^\circ \text{ and } f_y(x_0, y_0) = \tan 3^\circ$$

Then

$$\nabla f(x_0, y_0) = \langle -\tan 4^\circ, \tan 3^\circ \rangle \approx \langle -0.06993, 0.05241 \rangle,$$

somewhat NW.

$$\| \nabla f(x_0, y_0) \| \approx 0.08739.$$

Then $\theta \approx \arctan 0.08739 \approx 4.49.$
Extending to three variables:

**Definition.** The directional derivative of \( f(x, y, z) \) at the point \((a, b, c)\) and in the direction of the unit vector \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) is

\[
D_{\mathbf{u}} f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h}
\]

provided the limit exists.

The gradient of \( f(x, y, z) \) is the vector-valued function

\[
\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = \langle f_x, f_y, f_z \rangle
\]

provided all the partial derivatives exist.

**Theorem (6.4).** If \( f \) is a differentiable function of \( x, y, \) and \( z \) and \( \mathbf{u} = \) is any unit vector,

\[
D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.
\]

Here also \( \nabla f(a, b, c) \) points in the direction of maximum increase, which is \( \| \nabla f(a, b, c) \| \) at \((a, b, c)\).

If \( \mathbf{u} \) is in the tangent plane to the level surface \( f(x, y, z) = k \) at \((a, b, c)\), then

\[
D_{\mathbf{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{u} = 0 \implies \nabla f(a, b, c) \perp \mathbf{u}.
\]

**Theorem (6.5).** Suppose \( f(x, y, z) \) has continuous partial derivatives at \((a, b, c)\) and \( \nabla f(a, b, c) \neq \mathbf{0} \). Then \( \nabla f(a, b, c) \) is a normal vector to the tangent plane to the surface \( f(x, y, z) = k \) at the point \((a, b, c)\). Further, the equation of the tangent plane is

\[
f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0
\]

or

\[
\nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0.
\]
**EXAMPLE.** \( f(x, y, z) = x^2 - \frac{y}{z^2} \) and \( S \) is the level surface \( F(x, y, z) = 0 \).

(a) Find unit vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) in the directions of maximum increase at \((0, 0, 1)\) and \((1, 1, 1)\).

\[
\nabla f(x, y, z) = \left< 2x, -\frac{1}{z^2}, \frac{2y}{z^3} \right>.
\]

\[
\nabla f(0, 0, 1) = \left< 0, -1, 0 \right> = \mathbf{u}_1.
\]

\[
\nabla f(1, 1, 1) = \left< 2, -1, 2 \right>\text{ and } \|\left< 2, -1, 2 \right>\| = \sqrt{9} = 3 \implies \mathbf{u}_2 = \left< \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right>.
\]

(b) Find tangent planes to \( S \) at \((0, 0, 1)\) and \((1, 1, 1)\).

\[
f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0
\]

at \((0, 0, 1)\):

\[-1(y - 0) = 0 \quad \text{or} \quad y = 0.
\]

at \((1, 1, 1)\):

\[2(x - 1) - (y - 1) + 2(z - 1) = 0 \quad \text{or} \quad 2x - y + 2z - 3 = 0.
\]

(c) Find points on \( S \) where the normal vector is parallel to the \( x-y \) plane.

\[
\nabla f(x, y, z) \parallel x-y \text{ plane} \implies \nabla f(x, y, z) \perp \mathbf{k} (= \mathbf{u}) \implies
\]

\[
\nabla f(x, y, z) \cdot \mathbf{k} = \frac{2y}{z^3} = 0 \implies y = 0 \implies
\]

(since \( F(x, y, z) = x^2 - \frac{y}{z^2} = 0 \)) \( x^2 = 0 \implies x = 0.\)

Thus points of the form \((0, 0, z)\) where \( z \neq 0.\)

**MAPLE.** See graddirect(12.6).mw or graddirect(12.6).pdf
7. Extrema of Functions of Several Variables

The graph below is for the function

\[ z = xe^{-\frac{x^2}{2} - \frac{y^3}{3} + y}. \]

It appears to be a plain with a mountain and a valley.

**Definition.** We call \( f(a, b) \) a local maximum of \( f \) if there is an open disk \( R \) centered at \((a, b)\) for which \( f(a, b) \geq f(x, y) \) for all \((x, y) \in R\). \( f(a, b) \) is a local minimum of \( f \) if there is an open disk \( R \) centered at \((a, b)\) for which \( f(a, b) \leq f(x, y) \) for all \((x, y) \in R\). In either case, \( f(a, b) \) is called a local extremum of \( f \).

From the figure above, it appears that the surface has horizontal tangent planes at local extrema, provided such tangent planes exist. Also, recalling \( \nabla f(a, b) \) indicates the path of greatest increase and \(-\nabla f(a, b)\) the path of greatest decrease from \((a, b)\), \( \nabla f(a, b) = 0 \) if it exists.

**Definition.** The point \((a, b)\) is a critical point of \( f(x, y) \) if \((a, b)\) is in the domain of \( f \) and \( \nabla f(a, b) = 0 \) or does not exist.

**Theorem (7.1).** If \( f(x, y) \) has a local extremum at \((a, b)\), then \((a, b)\) is a critical point of \( f \).
Example (7.1). $z = xe^{-\frac{x^2}{2} - \frac{y^3}{3} + y}$.

$$f_x = e^{-\frac{x^2}{2} - \frac{y^3}{3} + y} + x(-x)e^{-\frac{x^2}{2} - \frac{y^3}{3} + y} = (1 - x^2)e^{-\frac{x^2}{2} - \frac{y^3}{3} + y}.$$  

$$f_y = x(-y^2 + 1)e^{-\frac{x^2}{2} - \frac{y^3}{3} + y}.$$  

Now  

$$f_x = 0 \iff x = \pm 1 \quad \text{and} \quad f_y = 0 \iff x = 0 \text{ or } y = \pm 1.$$  

Thus, for $\nabla f(a, b) = 0$,  

$$(x, y) = (1, 1), 1, -1), (-1, 1), \text{ or } (-1, -1).$$  

From the previous graph and the contour plot, we have a local maximum at $(1, 1)$ and a local minimum at $(-1, 1)$. At the other two points we have saddle points.

Notice the concentric “circles” near the local maximum and minimum and the hyperbolic-looking curves near saddle points.
Example. \( f(x, y) = ax^2 + bxy + cy^2 \).

\[
\begin{align*}
  f_x &= 2ax + by \quad \text{and} \quad f_y = bx + 2cy, \\
  f_{xx} &= 2a, \quad f_{xy} = b, \quad \text{and} \quad f_{yy} = 2c.
\end{align*}
\]

Clearly, \((0, 0)\) is a critical point (there may be others).

Let \( D = f_{xx}f_{yy} - f_{xy}^2 = 4ac - b^2 \).

For \( a \neq 0 \),

\[
  f(x, y) = a \left[ x^2 + \frac{b}{a}xy + \frac{c}{a}y^2 \right] = a \left[ \left( x + \frac{by}{2a} \right)^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right)y^2 \right] \implies 
  f(x, y) = a \left[ \left( x + \frac{b}{2a}y \right)^2 + \left( \frac{4ac - b^2}{4a^2} \right)y^2 \right].
\]

1) \( D > 0 \implies \) the expression in brackets is \( \geq 0 \).

a) \( a > 0 \implies \) local minimum at \((0, 0)\) since \( f(x, y) \geq 0 \) and \( f(0, 0) = 0 \) (upward opening elliptic paraboloid).

b) \( a < 0 \implies \) local maximum at \((0, 0)\) since \( f(x, y) \leq 0 \) and \( f(0, 0) = 0 \) (downward opening elliptic paraboloid).

2) \( D < 0 \implies \) saddle point at \((0, 0)\), no maximum or minimum, graph like that of \( z = x^2 - y^2 \).

3) \( D = 0 \implies f(x, y) = a \left( x + \frac{b}{2a}y \right)^2 \) and the graph is a parabolic cylinder.

Every point on the line \( x + \frac{b}{2a}y = 0 \) is a local maximum \((a < 0)\) or local minimum \((a > 0)\).
**Theorem (7.2 - 2nd Derivative Test).** Suppose $f(x, y)$ has continuous second-order derivatives in some open disk containing the point $(a, b)$ and that $\nabla f(a, b) = 0$. Define the discriminant $D$ for the points $(a, b)$ by

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$ 

(1) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local minimum at $(a, b)$.

(2) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local maximum at $(a, b)$.

(3) If $D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.

(4) If $D(a, b) = 0$, then no conclusion can be drawn.

**Example.** $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

$f_x = 6xy - 6x = 6x(y - 1) = 0 \iff x = 0$ or $y = 1$.

For $x = 0$, $f_y = 0 \implies 3y(y - 2) = 0 \implies y = 0$ or $y = 2$.

Thus $(0, 0)$ and $(0, 2)$ are critical points.

For $y = 1$, $f_y = 0 \implies 3x^2 - 3 = 3(x + 1)(x - 1) = 0 \implies x = \pm 1$.

Thus $(1, 1)$ and $(-1, 1)$ are critical points.

Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$, $f_{xy} = 6x \implies D = 36(y - 1)^2 - 36x^2$.

At $(0, 0)$: $D = 36$, $f_{xx} = -6 \implies$ local maximum with $f(0, 0) = 2$.

At $(0, 2)$: $D = 36$, $f_{xx} = 6 \implies$ local minimum with $f(0, 2) = -2$.

At $(\pm 1, 1)$: $D = -36 \implies$ saddle point with $f(\pm 1, 1) = 0$. 
**Problem (Page 892 #18).** \( f(x, y) = 2y(x + 2) - x^2 + y^4 - 9y^2. \)

\[ f_x = 2y - 2x \quad \text{and} \quad f_y = 2x + 4 + 4y^3 - 18y. \]

\[ \nabla f = \langle 0, 0 \rangle \implies 2y - 2x = 0 \quad \text{and} \quad 2x + 4 + 4y^3 - 18y = 0. \]

First equation \( \implies x = y. \) Substituting into second equation \( \implies \)

\[ 4x^3 - 16x + 4 = 0 \implies x \approx -2.1149, 0.2541, 1.8608. \]

Then critical points are \((-2.1149, -2.1149), (0.2541, 0.2541),\) and \((1.8608, 1.8608).\)

\[ f_{xx} = -2, \quad f_{yy} = 12y^2 - 18, \quad f_{xy} = 2 \implies D = -24y^2 + 32. \]

At \((-2.1149, -2.1149): \quad D \approx -75.348 \implies \text{saddle point}. \]

At \((0.2541, 0.2541): \quad D \approx 30.4504 \quad \text{and} \quad f_{xx} < 0 \implies \text{local maximum}. \]

At \((1.8608, 1.8608): \quad D \approx -51.1024 \implies \text{saddle point}. \]
Problem. The following data show the age and income for a small number of people. Use the linear model to predict the income of a 45-year-old and of an 80-year-old. Comment on how accurate you think the model is.

<table>
<thead>
<tr>
<th>Age</th>
<th>Income in $1000's</th>
<th>$ax + b$</th>
<th>Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>30</td>
<td>24$a$ + $b$</td>
<td>24$a$ + $b$ - 30</td>
</tr>
<tr>
<td>32</td>
<td>34</td>
<td>32$a$ + $b$</td>
<td>32$a$ + $b$ - 34</td>
</tr>
<tr>
<td>40</td>
<td>52</td>
<td>40$a$ + $b$</td>
<td>40$a$ + $b$ - 52</td>
</tr>
<tr>
<td>56</td>
<td>82</td>
<td>56$a$ + $b$</td>
<td>56$a$ + $b$ - 82</td>
</tr>
</tbody>
</table>

We want to find the line $y = ax + b$ such that the sum of the squares of the residuals is minimized. The residuals are the vertical distances from the points to the regression line.

We wish to minimize

$$f(a, b) = (24a + b - 30)^2 + (32a + b - 34)^2 + (40a + b - 52)^2 + (56a + b - 82)^2.$$

$$f_a = 48(24a + b - 30) + 64(32a + b - 34) + 80(40a + b - 52) + 112(56a + b - 82)$$

$$\implies f_a = 12672a + 304b - 16960$$

$$f_b = 2(24a + b - 30) + 2(32a + b - 34) + 2(40a + b - 52) + 2(56a + b - 82)$$

$$\implies f_b = 304a + 8b - 396$$

Solving $\nabla f = 0$ means solving

$$\begin{cases} 12672a + 304b = 16960 \\ 304a + 8b = 396 \end{cases}$$

to get $a \approx 1.71$ and $b \approx -15.37$.

Thus the regression line is $y = 1.71x - 15.37$. We have $y(45) = $61,580 and $y(80) = $121,430. This model has value only for $x$ between 24 and 56.
7. Extrema of Functions of Several Variables

**Definition.**

1. \( f(a, b) \) is the **absolute maximum** of \( f \) on the region \( R \) if \( f(a, b) \geq f(x, y) \) for all \( (x, y) \in R \). \( f(a, b) \) is the **absolute minimum** of \( f \) on the region \( R \) if \( f(a, b) \leq f(x, y) \) for all \( (x, y) \in R \).

2. A region \( R \in \mathbb{R}^2 \) is bounded if there is a disk that completely contains \( R \).

**Theorem (7.3 – Extreme Value Theorem).** Suppose \( f(x, y) \) is continuous on the closed, bounded region \( R \in \mathbb{R}^2 \). Then \( f \) has both an absolute minimum and maximum on \( R \). Further, an absolute extremum may only occur at a critical point in \( R \) or at a point on the boundary of \( R \).

**Practical Procedure:**

1. Find all critical points of \( f \) in \( R \).
2. Find maximum and minimum values of \( f \) on the boundary of \( R \).
3. Compare values.
EXAMPLE. Find the absolute maximum and minimum values of
\[ f(x, y) = 2 + 2x + 2y - x^2 - y^2 \]
on the triangular area bounded by \( x = 0, y = 0, \) and \( y = 9 - x. \)

\[ f_x = 2 - 2x \] and \( f_y = 2 - 2y. \)

\[ \nabla f = 0 \implies x = 1 \text{ and } y = 1 \implies (1, 1) \text{ is a critical point and } f(1, 1) = 4. \]

On \( y = 0: \) \( f(x, y) = 2 + 2x - x^2 = g(x). \)

\[ g'(x) = 2 - 2x = 0 \implies x = 1 \text{ and } g(1) = f(1, 0) = 3. \]

Also, \( f(0, 0) = 2 \) and \( f(9, 0) = -61. \)

On \( x = 0: \) \( f(x, y) = 2 + 2y - y^2 = g(y). \)

By symmetry, \( f(0, 1) = 3 \) and \( f(0, 9) = -61. \)

On \( y = 9 - x: \)

\[ f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -2x^2 + 18x - 61 = g(x). \]

\[ g'(x) = -4x + 18 = 0 \implies x = \frac{9}{2} \text{ is a critical point and } f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}. \]

Thus the absolute maximum is \( f(1, 1) = 4 \) and the absolute minimum is \( f(0, 9) = f(9, 0) = -61. \)
Example. A delivery company accepts only boxes whose length and girth (perimeter of a cross section) do not sum over 108 inches. Find the dimensions that provide the largest volume.

Let \( x, y, z, V \) represent length, width, height, and volume, respectively.

It should be clear that the volume is maximized when the sum of the length and girth is 108. Otherwise, one could get more volume by simply increasing the length. Thus

\[
x + 2y + 2z = 108 \implies 0 < x = 108 - 2y - 2z.
\]

\[
V = xyz \implies V(y, z) = (108 - 2y - 2z)yz = 108yz - 2y^2z - 2yz^2.
\]

Since \( x, y, z > 0 \), there are no boundary values.

\[
V_y = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z
\]

\[
V_z = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y
\]

\[
\nabla f = 0 \implies 108 - 4y - 2z = 0 \text{ and } 108 - 2y - 4z = 0 \implies \\
\begin{cases}
4y + 2z = 108 \\
2y + 4z = 108
\end{cases} \implies x = y = 18.
\]

\[
V(18, 18) = 11, 664.
\]

Thus \( (18, 18) \) gives the maximum volume and the dimensions are \( 36 \times 18 \times 18 \).

Maple. See extrema(12.7).mw or extrema(12.7).pdf