CHAPTER 4

Second-Order Linear Equations

1. Introduction: The Mass-Spring Oscillator

A horizontal mass-spring system

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end, as shown in the figure below. Devise a differential equation that governs the motion of this oscillator, taking into account the forces acting on it due to the spring elasticity, damping friction, and possible external influences. When the spring above is unstretched and the inertial mass m is still, the system is in equilibrium. We let $y(t)$ be amount of displacement from the equilibrium position at time t. When the spring is stretched or compressed, it exerts a force that resists the displacement and is usually directly proportional to the displacement.

By Hooke's Law,

$$F_{spring} = -ky,$$

where k is a stiffness constant. Hooke's Law is only valid for sufficiently small displacements.

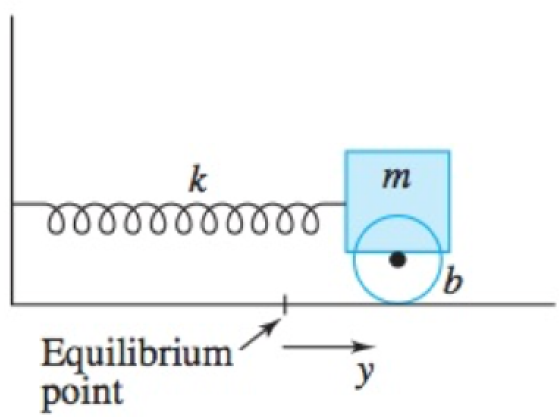


Figure 4.1 Damped mass–spring oscillator

Mechanical systems also experience friction and for vibrational motion is proportional to velocity:

$$F_{friction} = -b\frac{dy}{dt} = -by',$$

where $b \geq 0$ is the damping coefficient. Any other forces on the oscillator are regarded as external to the system. We lump all such forces together into a single, known fuction $F_{ext(t)}$. Then the DE for the mass-spring oscillator is

$$my'' = -ky - by' + F_{ext}$$

or

$$my'' + by' + ky = F_{ext}.$$

If $b = F_{ext}(t) = 0$, this DE becomes the equation of simple harmonic motion

$$my'' + ky = 0,$$

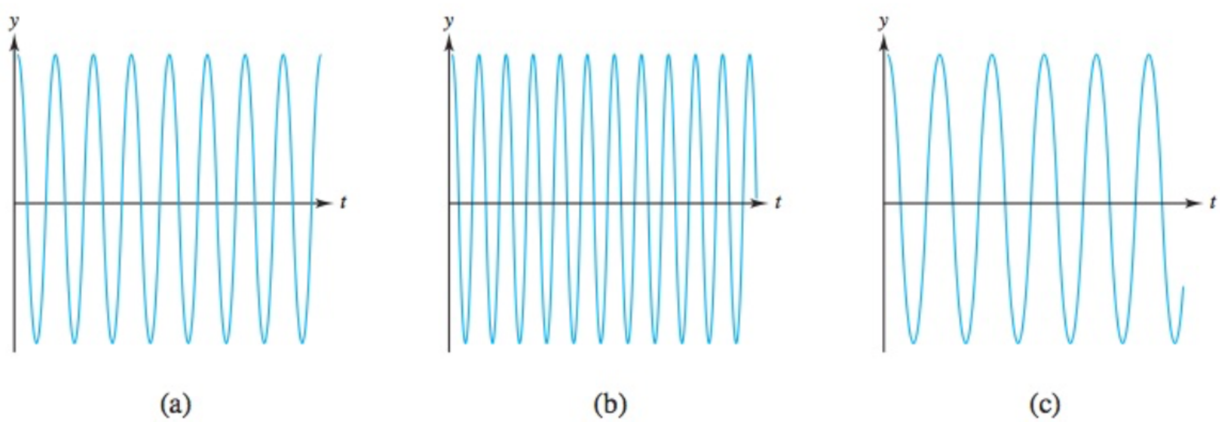
or

$$y'' + \beta^2 y = 0, \qquad \beta = \sqrt{\frac{k}{m}},$$

which has a solution of the form

$$y(t) = \cos\beta t.$$

The (idealized) mass-spring motions would be perpetual vibrations like the ones depicted below.



(a) Sinusoidal oscillation, (b) stiffer spring, and (c) heavier mass

These vibrations resemble sinusoidal functions, with their amplitude depending on the initial displacement and velocity. The frequency of the oscillations increases for stiffer springs but decreases for heavier masses.

DEFINITION. Any DE of the form

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

where $a(t)$, $b(t)$, $c(t)$, and $g(t)$ are continuous on some interval I, $a(t) \neq 0$ on I, is a <u>second-order linear DE</u>.

Any DE of the form

$$(*) \quad ay'' + by' + cy = 0$$

where a, b, and c are constants, $a \neq 0$, is a <u>second-order homogeneous linear DE with constant coefficients</u>.

<u>Goal</u>: To solve $(*)$ for all possible values of a, b, and c.

For understanding the math procedures and theory for solving $(*)$, we will consistently compare it with mass-spring paradigm:

$$[\textbf{inertia}] \times y'' + [\textbf{damping}] \times y' + [\textbf{stiffness}] \times y = F_{ext}.$$

2. Homogeneous Linear Equations: The General Solution

EXAMPLE.

(1) $y'' - y' = 0$.

Let $x = y'$. Then the DE becomes

$$x' - x = 0,$$

a first-order linear DE with a specific, not general, solution

$$x = e^t.$$

Then

$$y' = e^t \Longrightarrow$$

$$y = \int e^t \, dt = e^t$$

is a solution.

(2) $y'' + 6y' = 0$.

Let $x = y' \Longrightarrow x' + 6x = 0$. This has $x = e^{-6t}$ as a specific solution, so

$$y' = e^{-6t} \Longrightarrow y = -\frac{1}{6}e^{-6t}$$

is a specific solution. By checking,

$$y = e^{-6t}$$

is a solution. $\square$

Could it be that all solutions of $(*)$ are of the form Ce^{rt} for some value of r. We investigate.

Suppose, for some r, $y = e^{rt}$ is a solution of

$$(*) \quad a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

Then

$$\begin{gathered} ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \iff \text{(if and only if)} \\ (ar^2 + br + c)e^{rt} = 0 \iff \\ \underbrace{ar^2 + br + c = 0}_{\text{auxiliary or characteristic equation associated with } (*)} \qquad \iff \\ \underbrace{r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}_{\text{auxiliary orcharacteristic roots}} \end{gathered}$$

Let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note, if the discriminant $\Delta = b^2 - 4ac = 0$, then $r_1 = r_2 = -\dfrac{b}{2a}$, giving the characteristic equation a <u>double</u> or <u>repeated root</u> of multiplicity 2.

We have:

Lemma. *e^{rt} is a solution of $(*)$ $\iff$ r is a root of $ar^2 + br + c = 0$.*

Note. The same is true for Ce^{rt}.

But are there still other solutions?

Note that

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 r_2 = \frac{c}{a}.$$

Then

$$\begin{gathered} ay'' + by' + cy = 0 \iff \\ y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \iff \\ y'' - (r_1 + r_2)y' + r_1 r_2 y = 0 \iff \\ (y' - r_2 xy)' - r_1(y' - r_2 y) = 0. \end{gathered}$$

Now let $x = y' - r_2 y$. Then

$$x' - r_1 x = 0,$$

which has the general solution $x = k_1 e^{r_1 t}$ for $k_1 \in \mathbb{R}$. Substituting this solution into $x = y' - r_2 y$,

$$\begin{gathered} y' - r_2 y = k_1 e^{r_1 t} \iff \\ e^{-r_2 t} y' - r_2 e^{-r_2 t} y = k_1 e^{r_1 t} e^{-r_2 t} \iff \\ (\#) \quad \frac{d}{dt}\Big[e^{-r_2 t} y\Big] = k_1 e^{(r_1 - r_2)t}. \end{gathered}$$

Suppose $r_1 \neq r_2$. Then, integrating,

$$e^{-r_2 t} y = \int k_1 e^{(r_1 - r2)t}\, dt + k_2 = \frac{k_1}{r_1 - r_2} e^{(r_1 - r_2)t} + k_2 \iff$$

$$y(t) = \frac{k_1}{r_1 - r_2} e^{r_1 t} + k_2 e^{r_2 t},$$

with k_1 and k_2 arbitrary constants.

Letting $c_1 = \dfrac{k_1}{r_1 - r_2}$ and $c_2 = k_2$,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is a solution of $(*)$ for all values of c_1 and c_2.

But what does this mean if r_1 and/or r_2 is complex? Also, what about the case of $r_1 = r_2$?

$\boxed{\text{Case 1}}$: r_1 and r_2 real, $r_1 \neq r_2$.

For this case, we have established that all solutions of $(*)$ are of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Also, any function written in the above form is a solution of

$$y'' - (r_1 + r_2) y' + r_1 r_2 y = 0$$

since

$$(c_1 e^{r_1 t} + c_2 e^{r_2 t})'' - (r_1 + r_2)(c_1 e^{r_1 t} + c_2 e^{r_2 t})' + r_1 r_2 (c_1 e^{r_1 t} + c_2 e^{r_2 t}) =$$

$$c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t} - c_1 r_1^2 e^{r_1 t} - c_2 r_1 r_2 e^{r_2 t} - c_1 r_1 r_2 e^{r_1 t} \\ - c_2 r_2^2 e^{r_2 t} + c_1 r_1 r_2 e^{r_1 t} + c_2 r_1 r_2 e^{r_2 t} = 0.$$

We say

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is a <u>linear combination</u> of the functions $e^{r_1 t}$ and $e^{r_2 t}$.

Definition. For any two vectors $\mathbf{v}$ and $\mathbf{w}$,

$$c_1\mathbf{v} + c_2\mathbf{w}$$

is a <u>linear combination</u> of $\mathbf{v}$ and $\mathbf{w}$ where c_1 and c_2 are <u>scalars</u>.

Note. Functions can be considered as vectors, with the real numbers the scalars.

Definition. Two vectors are <u>linearly dependent</u> (l.d.) if they are proportional to each other.

We have

$$\mathbf{v} \text{ and } \mathbf{w} \text{ are l.d.} \iff \mathbf{w} \propto \mathbf{v} \iff \mathbf{w} = -\frac{c_1}{c_2}\mathbf{v} \iff$$
$$c_2\mathbf{w} = -c_1\mathbf{v} \iff c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{0}, c_2 \neq 0.$$

Corollary. *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are l.d.* $\iff$ *there exist scalars* c_1 *and* c_2*, not both 0, such that* $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{0}$*.*

Example. $f(t) = e^{4t}$ and $g(t) = 6e^{4t}$.

$$g(t) = 6f(t) \Longrightarrow -6f(t) + g(t) = 0.$$

Definition. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are <u>linearly independent</u> (l.i.) if they are not proportional to each other.

Corollary. *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are l.i.* $\iff$

$$\Big[c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{0} \Longrightarrow c_1 = c_2 = 0\Big].$$

EXAMPLE. $f(t) = e^t$ and $g(t) = \sin t$.

These are clearly l.i., since if $f(t) = kg(t)$,

$$\Big[t = 1 \Longrightarrow e = k \sin 1 \Longrightarrow k \approx 3.23\Big] \text{ and}$$

$$\Big[t = 2 \Longrightarrow e^2 = k \sin 2 \Longrightarrow k \approx 8.126\Big],$$

giving a contradiction.

DEFINITION. For a second-order homogeneous linear DE, two l.i. solutions are called a fundamental set of solutions.

THEOREM (Distinct Real Characteristic Roots & General Solution). *If the characteristic equation for*

$$(*) \quad ay'' + by' + cy = 0$$

has the distinct real roots r_1 and r_2, then $e^{r_1 t}$ and $e^{r_2 t}$ are linearly independent solutions of $(*)$ on the interval $(-\infty, \infty)$. Furthermore,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where c_1 and c_2 are arbitrary constants, is a general solution on $(-\infty, \infty)$.

EXAMPLE.

(1) Solve $y'' = 5y' + 24y$. This is equivalent to

$$y'' - 5y' - 24y = 0.$$

For the characteristic equation,

$$r^2 - 5r - 24 = (r - 8)(r + 3) = 0 \iff r = 8 \text{ or } r = -3.$$

Then the general solution is

$$y(t) = c_1 e^{8t} + c_2 e^{-3t}.$$

(2) Solve $y'' + 3y' = 0$.

For the characteristic equation,

$$r^2 + 3r = r(r+3) = 0 \iff r = -3 \text{ or } r = 0.$$

Then the general solution is

$$y(t) = c_1 e^{-3t} + c_2 e^{0t} = c_1 e^{-3t} + c_2.$$

Initial Value Problems

Suppose in $(*)$, $a = 1$ and $b = c = 0$, yielding the DE $y'' = 0$. Then the graph of $y(t)$ is simply a straight line and is uniquely determined by specifying a point on the line,

$$y(t_0) = Y_0$$

and the slope of the line,

$$y'(t_0) = Y_1.$$

The next theorem states that this idea transfers to the more general equation $(*)$.

THEOREM (1 — Existence and Uniqueness: Homogeneous Case). *For any real numbers $a(\neq 0)$, b, c, t_0, Y_0, and Y_1, there exists a unique solution to the IVP*

$$(**) \qquad ay'' + by' + cy = 0, \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

The solution is valid for all $t \in (-\infty, +\infty)$.

But further,

THEOREM (2 — Representation of Solutions to an IVP). *If $y_1(t)$ and $y_2(t)$ are any two solutions of $(**)$ that are l.i. (i.e., a fundamental set of solutions) on $(-\infty, +\infty)$, then unique constants c_1 and c_2 can always be found so that $c_1 y_1(t) + c_2 y_2(t)$ satisfies the IVP $(**)$ on $(-\infty, +\infty)$.*

The proof of this theorem needs the following technical lemma.

DEFINITION (Wronksian). The function

$$W(t) = W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1 y_2' - y_1' y_2$$

where y_1 and y_2 are any differentiable functions of t, is called the Wronksian of y_1 and y_2.

LEMMA (1— A condition for Linear Dependence of Solutions). *For any real numbers $a(\neq 0)$, b, and c, if $y_1(t)$ and $y_2(t)$ are any two solutions of $(*)$ on $(-\infty, +\infty)$ and if the equality*

$$(\#) \qquad W[y_1, y_2](\tau) = 0$$

holds at any point τ, then y_1 and y_2 are l.d. on $(-\infty, +\infty)$.

PROOF.

(1) If $y_1(\tau) \neq 0$, let $\kappa = \dfrac{y_2(\tau)}{y_1(\tau)}$ and consider the solution to $(*)$ given by $y(t) = \kappa y_1(t)$. For the IC of $(**)$, we have, with the second equality following from $(\#)$,

$$y(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1(\tau) = y_2(\tau), \quad y'(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1'(\tau) = y_2'(\tau),$$

which are the same as those of y_2, so by uniqueness $y_2(t) = \kappa y_1(t)$.

(2) If $y_1(\tau) = 0$ but $y_1'(\tau) \neq 0$, then $(\#)$ implies $y_2(\tau) = 0$. Let $\kappa = \dfrac{y_2'(\tau)}{y_1'(\tau)}$. Again, for $y(t) = \kappa y_1(t)$, for the IC of $(**)$, we have

$$y(\tau) = \frac{y_2'(\tau)}{y_1'(\tau)} y_1(\tau) = 0 = y_2(\tau), \quad y'(\tau) = \frac{y_2'(\tau)}{y_1'(\tau)} y_1'(\tau) = y_2'(\tau),$$

which are the same as those of y_2, so by uniqueness $y_2(t) = \kappa y_1(t)$.

(3) If $y_1(\tau) = y_1'(\tau) = 0$, then $y_1(t)$ is a solution to $(*)$ satisfying the IC $y_1(\tau) = y_1'(\tau) = 0$. But $y(t) \equiv 0$ is the unique solution to this IVP. Thus $y_1(t) \equiv 0$, and is then a constant multiple of $y_2(t)$. □

We now prove Theorem 2:

PROOF.

We already know $y(t) = c_1y_1(t) + c_2y_2(t)$ is a solution of $(*)$.For $(**)$, we must show

$$y(t_0) = c_1y_1(t_0) + c_2y_2(t_0) = Y_0$$

and

$$y'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0) = Y_1$$

But these are linear equations where the solutions are

$$c_1 = \frac{Y_0y_2'(t_0) - Y_1y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} \text{ and } c_2 = \frac{Y_1y_1't_0) - Y_0y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)},$$

provided the denominators are nonzero, which is the case here. □

EXAMPLE. Solve $2y'' + 7y' + 5y = 0, \quad y(0) = -2, \quad y'(0) = 4.$

For the characteristic equation,

$$2r^2 + 7r + 5 = (2r + 5)(r + 1) = 0 \iff r = -\frac{5}{2} \text{ or } r = -1.$$

Then the general solution is

$$y(t) = c_1e^{-\frac{5}{2}t} + c_2e^{-t}.$$

Then

$$y'(t) = -\frac{5}{2}c_1e^{-\frac{5}{2}t} - c_2e^{-t}.$$

From the initial conditions, we have

$$y(0) = c_1 + c_2 = -2$$
$$y'(0) = -\frac{5}{2}c_1 - c_2 = 4$$

Solving for c_1 and c_2 using elimination,

$$-\frac{3}{2}c_1 = 2 \Longrightarrow c_1 = -\frac{4}{3} \Longrightarrow c_2 = -\frac{2}{3}.$$

This yields a particular solution of

$$y(t) = -\frac{4}{3}e^{-\frac{5}{2}t} - \frac{2}{3}e^{-t}.$$

Case 2 : r_1 and r_2 real, $r_1 = r_2$.

In solving

$$(*) \quad ay'' + by' + cy = 0,$$

we return to

$$(\#) \quad \frac{d}{dt}\left[e^{-r_2 t}y\right] = k_1 e^{(r_1 - r_2)t}.$$

Since $r_1 = r_2$, this becomes, after renamimg k_1 as c_2,

$$\frac{d}{dt}\left[e^{-r_2 t}y\right] = c_2.$$

Integrating, we have

$$e^{-r_2 t}y = c_2 t + c_1 \Longrightarrow y = (c_1 + c_2 t)e^{r_2 t} \Longrightarrow \text{(letting } r = r_2\text{)}$$
$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Theorem (Repeated Characteristic Roots & General Solution). *If the characteristic equation for*

$$(*) \quad ay'' + by' + cy = 0$$

has the real repeated root r, then e^{rt} and te^{rt} are linearly independent solutions of $(*)$ on the interval $(-\infty, \infty)$. Furthermore,

$$y(t) = c_1 e^{rt} + c_2 t e^{rt},$$

where c_1 and c_2 are arbitrary constants, is a general solution on $(-\infty, \infty)$.

Example.

(1) Solve $3y'' - 18y' + 27y = 0$.

This is equivalent to

$$y'' - 6y' + 9y = 0.$$

For the characteristic equation,

$$r^2 - 6r + 9 = (r-3)^2 = 0 \iff r = 3, 3.$$

Then the general solution is

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

(2) Solve $y'' + 8y' + 16y = 0, \quad y(0) = 4, \quad y'(0) = 0$.

For the characteristic equation,

$$r^2 + 8r + 16 = (r+4)^2 = 0 \iff r = -4, -4.$$

Then the general solution is

$$y(t) = c_1 e^{-4t} + c_2 t e^{-4t}.$$

Then

$$y'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}.$$

From the initial conditions, we have

$$y(0) = c_1 = 4$$
$$y'(0) = -4c_1 + c_2 = 0$$

Solving for c_1 and c_2 using substitution,

$$c_2 = 4c_1 = 16.$$

This yields a particular solution of

$$y(t) = 4e^{-4t} + 16te^{-4t}.$$

A Question:

For second-order homogeneous linear DE's, do linear combinations of solutions always give another solution?

Suppose f'' exists on an interval $J = (\alpha, \beta)$. We say f is twice differentiable on J. Further, suppose $a(t)$, $b(t)$, and $c(t)$ are continuous on J. We define the differential operator L by

$$L[f] = af'' + bf' + cf.$$

This is a function whose domain is the set of twice differentiable functions on J. The range is a subset of the continuous functions on J.Letting $D = \dfrac{d}{dt}$ and $D^2 = \dfrac{d^2}{dt^2}$, we have

$$L = aD^2 + bD + c,$$

yielding

$$\begin{aligned} L[f] &= (aD^2 + bD + c)f \\ &= aD^2 f + bDf + cf \\ &= af'' + bf' + cf \end{aligned}$$

With this notation and a, b, and c constants,

$$(*) \text{ becomes } L[y] = 0.$$

L is a linear differential operator in that, for f and g twice differentiable and γ and λ continuous coefficients,

$$\begin{gathered} L[\gamma f] = \gamma L[f] \text{ and} \\ L[f+g] = L[f] + L[g], \text{ or equivalently,} \\ L[\gamma f + \lambda g] = \gamma L[f] + \lambda L[g]. \end{gathered}$$

PROOF.

$$\begin{aligned} L[\gamma f + \lambda g] &= a(\gamma f + \lambda g)'' + b(\gamma f + \lambda g)' + c(\gamma f + \lambda g) \\ &= a\gamma f'' + a\lambda g'' + b\gamma f' + b\lambda g' + c\gamma f + c\lambda g \\ &= \big[\gamma(af'' + bf' + cf)\big] + \big[\lambda(ag'' + bg' + cg)\big] \\ &= \gamma L[f] + \lambda L[g]. \end{aligned}$$

□

NOTE.

(1) Integration is also a linear operator.

(2) sin, cos, tan, exp, and ln are not linear operators, e.g.,

$$\sin(a+b) \neq \sin a + \sin b.$$

THEOREM (3 — Superposition Principal). *If $y_1(t)$ and $y_2(t)$ are solutions of*

$$(*) \quad ay'' + by' + cy = 0$$

on $(-\infty, \infty)$, then so is the linear combination

$$c_1 y_1(t) + c_2 y_2(t),$$

regardless of the values of the constants c_1 and c_2.

PROOF. Let $y_1(t)$ and $y_2(t)$ be solutions of $L[y] = 0$ on $(-\infty, \infty)$ where $L = aD^2 + bD + c$. Then $L[y_1(t)] = 0$ and $L[y_2(t)] = 0$ for all $-\infty < t < \infty$. Thus for all constants c_1 and c_2,

$$L[c_1 y_1(t) + c_2 y_2(t)] = c_1 L[y_1(t)] + c_2 L[y_2(t)] = c_1 \cdot 0 + c_2 \cdot 0 = 0,$$

so $c_1 y_1(t) + c_2 y_2(t)$ is also a solution. □

PROBLEM (Page 166 # 40).

Solve $y''' - 7y'' + 7y' + 15y = 0$.

SOLUTION. We informally expand on our method. The associated characteristic equation is

$$r^3 - 7r^2 + 7r + 15 = 0.$$

We use synthetic division (Horner's method) to find the roots.

$$\begin{array}{r|rrrr} -1 & 1 & -7 & 7 & 15 \\ & & -1 & 8 & -15 \\ \hline & 1 & -8 & 15 & |\underline{0} \end{array}$$

Thus we have $r = -1$ as a root with the remaining roots being roots of

$$r^2 - 8r + 15 = (r-5)(r-3) = 0,$$

namely $r = 5$ and $r = 3$. Thus e^{-t}, e^{5t}, and e^{3t} are linearly independent solutions yielding a general solution of

$$y(t) = c_1 e^{-t} + c_2 e^{5t} + c_3 e^{3t}.$$

□

3. Auxiliary Equations with Complex Roots

Case 3: r_1 and r_2 are conjugate complex roots.

The solution to the characteristic equation is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ with } b^2 - 4ac < 0.$$

Then

$$r = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a} \text{ with } 4ac - b^2 > 0.$$

Let

$$\alpha = -\frac{b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

Then, renaming r_1 and r_2, the two conjugate complex roots of the characteristic equation are

$$r = \alpha + i\beta \quad \text{and} \quad \overline{r} = \alpha - i\beta.$$

Also,

$$\alpha = \Re r = \Re \overline{r}, \quad \beta = \Im r, \quad \text{and } -\beta = \Im \overline{r}.$$

We first consider the simple harmonic oscillator we began with:

$$(**) \quad y'' + \beta^2 y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0, \quad \beta \neq 0.$$

It is clear that $y(t) \equiv 0$ is a solution of $(**)$. But are there others?

Suppose $u(t)$ is any solution of $(**)$. Then

$$u''(t) + \beta^2 u(t) = 0, \quad u(t_0) = 0, \quad u'(t_0) = 0.$$

So

$$\begin{gathered}
2u'(t)\big[u''(t) + \beta^2 u(t)\big] = 0 \Longrightarrow \\
\int_{t_0}^{t} 2u'(s)\big[u''(s) + \beta^2 u(s)\big]\, ds = \int_{t_0}^{t} 0\, ds \Longrightarrow \\
\Big\{[u'(s)]^2 + \beta^2 [u(s)]^2\Big\}\Big|_{t_0}^{t} = 0 \Longrightarrow \\
[u'(t)]^2 + \beta^2[u(t)]^2 - [u'(t_0)]^2 - \beta^2[u(t_0)]^2 = 0 \Longrightarrow \\
\text{(from IC)} \quad [u'(t)]^2 + \beta^2[u(t)]^2 = 0 \Longrightarrow \\
u(t) = 0.
\end{gathered}$$

Thus we have

LEMMA (1). *Let t_0, β be real numbers, where $\beta \neq 0$. Then the IVP*

$$(**) \quad y'' + \beta^2 y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0$$

has the unique solution $y(t) \equiv 0$ on $(-\infty, \infty)$.

Now consider

$$(***) \quad y'' + \beta^2 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \beta \neq 0.$$

Suppose $y(t)$ and $\widetilde{y}(t)$ are two solutions of $(***)$. Consider

$$u(t) = y(t) - \widetilde{y}(t).$$

By Superposition, $u(t)$ is a solution of

$$y'' + \beta^2 y = 0.$$

Also,

$$u(t_0) = y(t_0) - \widetilde{y}(t_0) = y_0 - y_0 = 0$$

and

$$u'(t_0) = y'(t_0) - \widetilde{y}\,'(t_0) = y_1 - y_1 = 0 \Longrightarrow$$

$u(t)$ is a solution of $(**)$, so $u(t) \equiv 0$. Then $y(t) = \widetilde{y}(t)$. We have proven

LEMMA (2). *Let t_0, y_0, y_1, and β be real numbers, where $\beta \neq 0$. If the IVP*

$$(***) \quad y'' + \beta^2 y = 0, \quad x(t_0) = y_0, \quad y'(t_0) = y_1$$

has a solution on $(-\infty, \infty)$, then it is unique.

Question: Does $(***)$ always have a solution?

For $\beta \neq 0$, we have an undamped mass-spring system which suggests repetitious behavior.

Suppose $y(t) = \cos\beta t$. Then

$$y'' + \beta^2 y = \frac{d^2}{dt^2}\cos\beta t + \beta^2\cos\beta t = -\beta^2\cos\beta t + \beta^2\cos\beta t = 0,$$

so $y(t) = \cos\beta t$ is a solution of $y'' + \beta^2 y = 0$.

Similarly, $y(t) = \sin\beta t$ is also a solution.

Then, by superposition, anything of the form

$$(\#\#) \quad y(t) = c_1 \cos \beta t + c_2 \sin \beta t$$

is also a solution of $y'' + \beta^2 y = 0$.

But do these solutions satisfy the IC of $(***)$?

For $(\#\#)$ to be a solution of $(***)$, we need, noting

$$y'(t) = -c_1 \beta \sin \beta t + c_2 \beta \cos \beta t,$$

$$y(t_0) = c_1 \cos \beta t_0 + c_2 \sin \beta t_0 = y_0 \text{ and}$$
$$y'(t_0) = -c_1 \beta \sin \beta t_0 + c_2 \beta \cos \beta t_0 = y_1$$

would have to have a solution for c_1 and c_2. In fact,

$$c_1 = y_0 \cos \beta t_0 - \frac{y_1}{\beta} \sin \beta t_0 \text{ and}$$

$$c_2 = y_0 \sin \beta t_0 + \frac{y_1}{\beta} \cos \beta t_0.$$

Thus

$$y(t) = c_1 \cos \beta t + c_2 \sin \beta t$$

with c_1 and c_2 as above is a solution of $(***)$. By Lemma 2, it is the only solution. We have proven

THEOREM (Simple Harmonic Oscillator and Solutions).

For any real numbers y_1, y_2, β, where $\beta \neq 0$, there exists a unique solution of

$$(***) \quad y'' + \beta^2 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

This solution can be expressed in the form

$$y(t) = c_1 \cos \beta t + c_2 \sin \beta t,$$

where the constants c_1 and c_2 are uniquely determined by y_0 and y_1.

We return to

$$(*) \quad ay'' + by' + cy = 0,$$

which we rewrite as

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0.$$

Since we are assuming $(*)$ has characteristic roots

$$r = \alpha + i\beta \text{ and } \overline{r} = \alpha - i\beta,$$

we have

$$\frac{b}{a} = -(r + \overline{r}) = -2\alpha$$

and

$$\frac{c}{a} = r \cdot \overline{r} = (\alpha + i\beta) \cdot (\alpha - i\beta) = \alpha^2 + \beta^2.$$

Then

$$(*') \quad y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$$

has the same solutions as $(*)$.

If $\beta = 0$, we have repeated roots, and from the previous theorem,

$$y(t) = c_1 e^{\alpha t} + c_2 t e^{\alpha t}$$

is a general solution $\Longrightarrow e^{\alpha t}$ is a solution.

Although $e^{\alpha t}$ is not a solution of $(*')$ since $\beta \neq 0$, every solution $y(t)$ for $(*')$ can be written as

$$y(t) = e^{\alpha t} x(t)$$

where

$$x(t) = \frac{y(t)}{e^{\alpha t}}.$$

Substituting $y(t) = e^{\alpha t}x(t)$ into $(*')$,

$$y'' - 2\alpha y' + (\alpha^2 + \beta^2)y =$$

$$\frac{d^2}{dt^2}(e^{\alpha t}x) - 2\alpha\frac{d}{dt}(e^{\alpha t}x) + (\alpha^2 + \beta^2)(e^{\alpha t}x) =$$

$$\frac{d}{dt}[e^{\alpha t}x' + \alpha e^{\alpha t}x] - 2\alpha(e^{\alpha t}x' + \alpha e^{\alpha t}x) + \alpha^2 e^{\alpha t}x + \beta^2 e^{\alpha t}x =$$

$$e^{\alpha t}x'' + \alpha e^{\alpha t}x' + \alpha e^{\alpha t}x' + \alpha^2 e^{\alpha t}x - 2\alpha e^{\alpha t}x' + \alpha^2 e^{\alpha t}x - 2\alpha^2 e^{\alpha t}x + \beta^2 e^{\alpha t}x =$$

$$e^{\alpha t}x'' + \beta^2 e^{\alpha t}x =$$

$$e^{\alpha t}(x'' + \beta^2 x) = 0 \iff$$

$$x'' + \beta^2 x = 0.$$

Thus, every solution of $(*')$ can be expressed as $y(t) = e^{\alpha t}x(t)$ where $x(t)$ is a solution of $x'' + \beta^2 x = 0$.

THEOREM (Complex Conjugate Roots and General Solution).

If the characteristic equation for

$$(*) \quad ay'' + by' + cy = 0$$

has the complex roots $\alpha \pm i\beta$ $(\beta \neq 0)$, *then* $e^{\alpha t}\cos\beta t$ *and* $e^{\alpha t}\sin\beta t$ *are l.i. solutions of* $(*)$ *on* $(-\infty, \infty)$. *A general solution on* $(-\infty, \infty)$ *is*

$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t,$$

where c_1 *and* c_2 *are arbitrary constants.*

EXAMPLE.

(1) Solve

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

The associated characteristic equation

$$r^2 + 2r + 5 = 0$$

has roots

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

Let $\alpha = -1$, $\beta = 2$. Then

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

is the general solution and

$$y'(t) = -2c_1 e^{-t} \sin 2t - c_1 e^{-t} \cos 2t + 2c_2 e^{-t} \cos 2t - c_2 e^{-t} \sin 2t.$$

Then

$$y(0) = c_1 = 1 \text{ and}$$
$$y'(0) = -c_1 + 2c_2 = -1 + 2c_2 = 1 \implies c_2 = 1.$$

Thus the solution to the IVP is

$$y(t) = e^{-t} \cos 2t + e^{-t} \sin 2t.$$

Referring back to the unforced mass-spring oscillator

$$[\textbf{inertia}] \times y'' + [\textbf{damping}] \times y' + [\textbf{stiffness}] \times y = 0$$

or

$$my''(t) + by'(t) + ky(t) = 0,$$

the exponential decay rate is $\alpha = -\dfrac{b}{2m}$ and the angular frequency is $\beta = \dfrac{\sqrt{4mk - b^2}}{2m}$. Since $b = 2 < \sqrt{4mk} = \sqrt{20}$, this is an example of an under-damped oscillator.

(2) Solve

$$y'' + 2y' + 10x = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

The associated characteristic equation

$$r^2 + 2r + 10 = 0$$

has roots

$$r = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i.$$

Let $\alpha = -1$, $\beta = 3$. Then

$$y(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t$$

is the general solution and

$$y'(t) = -3c_1 e^{-t} \sin 3t - c_1 e^{-t} \cos 3t + 3c_2 e^{-t} \cos 3t - c_2 e^{-t} \sin 3t.$$

Then

$$y(0) = c_1 = 1 \text{ and}$$

$$y'(0) = -c_1 + 3c_2 = -1 + 3c_2 = 0 \Longrightarrow c_2 = \frac{1}{3}.$$

Thus the solution to the IVP is

$$y(t) = e^{-t} \cos 3t + \frac{1}{3} e^{-t} \sin 3t.$$

Maple. See oscillate.mw or oscillate.pdf.

4. Nonhomogeneous Equations: The Method of Undetermined Coefficients

A second-order nonhomogeneous linear DE with constant coefficients is an equation of the form $ay'' + by' + cy = f(t)$ where $a \neq 0$ and f is any function other than the zero function.

EXAMPLE. $y'' - y = t$.

By inspection, $y = -t$ is a solution, called a particular solution.

THEOREM (Solutions of Second-Order Nonhomogeneous Equations). *A general solution of*

$$(*) \quad ay'' + by' + cy = f(t)$$

on an interval J is

$$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is a general solution of

$$ay'' + by' + cy = 0$$

and $y_p(t)$ is any particular solution of $(*)$ on J.

PROOF. We rewrite $(*)$ as $L[y] = f(t)$, and let $y_p(t)$ be any particular solution of $(*)$ on J (i,e., for each $t \in J$).

Suppose $y(t)$ is some other solution of $(*)$ on $J \Longrightarrow L[y] = f(t)$. Then

$$L[y(t) - y_p(t)] = L[y(t)] - L[y_p(t)] = f(t) - f(t) = 0,$$

so $y(t) - y_p(t)$ is a solution of the associated homogeneous equation $L[y] = 0$.

Let $y_h(t) = y(t) - y_p(t)$ (h for homogeneous)$\Longrightarrow$

$$y(t) = y_h(t) + y_p(t).$$

Thus any solution of $(*)$ can be written in this form.

On the other hand,

$$L[y_h(t) + y_p(t)] = L[y_h(t)] + L[y_p(t)] = f(t) + 0,$$

so anything of the form $y_h(t) + y_p(t)$ is a solution of $(*)$. $\square$

Finding Particular Solutions – Method of Undetermined Coefficients

This method is used when $f(t)$ is an l.c. of

(1) exponentials (e^{rt})

(2) polynomials

(3) sin or cos

(4) products of the above

EXAMPLE.

(1) Solve $y'' - y = 3e^{-2t}$.

$$r^2 - 1 = (r+1)(r-1) = 0 \Longrightarrow r = \pm 1.$$
$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

For the particular solution:

$$L[y] = 3e^{-2t}.$$

We guess

$$L[Ae^{-2t}] = 3e^{-2t} \Longrightarrow$$
$$4Ae^{-2t} - Ae^{-2t} = 3e^{-2t} \Longrightarrow$$
$$3Ae^{-2t} = 3e^{-2t} \Longrightarrow A = 1$$

Then

$$y_p(t) = e^{-2t} \Longrightarrow$$
$$y(t) = c_1 e^t + c_2 e^{-t} + e^{-2t}.$$

(2) Solve $y'' + 3y' + 2y = 2t^2 + 3t + 1$.

$$r^2 + 3r + 2 = (r+1)(r+2) = 0 \Longrightarrow r = -1, -2.$$
$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

For the particular solution:

$$L[y] = 2t^2 + 3t + 1.$$

We guess

$$L[At^2 + Bt + C] = 2t^2 + 3t + 1 \Longrightarrow$$

$$2A + 3(2At + B) + 2(At^2 + Bt + C) =$$
$$2At^2 + (6A + 2B)t + (2A + 3B + 2C) = 2t^2 + 3t + 1 \Longrightarrow$$

$2A = 2 \Longrightarrow A = 1$

$6A + 2B = 3 \Longrightarrow 2B = -3 \Longrightarrow B = -\frac{3}{2}$

$2A + 3B + 2C = 1 \Longrightarrow 2C = \frac{7}{2} \Longrightarrow C = \frac{7}{4}$

Then

$$y_p(t) = t^2 - \frac{3}{2}t + \frac{7}{4} \Longrightarrow$$
$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + t^2 - \frac{3}{2}t + \frac{7}{4}.$$

(3) Solve $y'' - y' - y = \sin t$.

$$r^2 - r - 1 = 0 \Longrightarrow r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$$y_h(t) = c_1 e^{\frac{1+\sqrt{5}}{2}t} + c_2 e^{\frac{1-\sqrt{5}}{2}t}.$$

For the particular solution:

$$L[y] = \sin t.$$

We guess

$$L[A \sin t] = \sin t \Longrightarrow$$

$$-A \sin t - A \cos t - A \sin t = -A(2 \sin t + \cos t) = \sin t \Longrightarrow$$
$$-2A = 1 \text{ and } -A = 0.$$

But this is impossible. A better guess is

$$L[A \sin t + B \cos t] = \sin t \Longrightarrow$$

$$(-A \sin t - B \cos t) - (A \cos t - B \sin t) - (A \sin t + B \cos t) =$$
$$(-2A + B) \sin t + (-A - 2B) \cos t = \sin t \Longrightarrow$$
$$-2A + B = 1 \text{ and } -A - 2B = 0 \Longrightarrow$$
$$A = -2B \Longrightarrow 4B + B = 1 \Longrightarrow 5B = 1 \Longrightarrow$$
$$B = \frac{1}{5} \Longrightarrow A = -\frac{2}{5}.$$

Then

$$y_p(t) = -\frac{2}{5} \sin t + \frac{1}{5} \cos t \Longrightarrow$$
$$y(t) = c_1 e^{\frac{1+\sqrt{5}}{2}t} + c_2 e^{\frac{1-\sqrt{5}}{2}t} - \frac{2}{5} \sin t + \frac{1}{5} \cos t.$$

Method of Undetermined Coefficients – Advanced

EXAMPLE. Solve $y'' - y = 2e^{-t}$.

From Example 1,

$$y_h(t) = c_1e^t + c_2e^{-t}.$$

For the particular solution to

$$L[y] = 2e^{-t},$$

we guess

$$\begin{gathered} y(t) = Ae^{-t} \implies \\ y'(t) = -Ae^{-t} \implies \\ y''(t) = Ae^{-t} \implies \\ L[y] = Ae^{-t} - Ae^{-t} = 0 \neq 2e^{-t}. \end{gathered}$$

We are left without an equation to solve for A. The problem is that a term of the trial solution is similar to a solution of the homogeneous equation. The remedy is to multiply the trial solution by the smallest power of t so that is no longer the case. We guess

$$\begin{gathered} y(t) = Ate^{-t} \implies \\ y'(t) = Ae^{-t} - Ate^{-t} \implies \\ y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} \implies \\ L[y] = -2Ae^{-t} + Ate^{-t} - Ate^{-t} = -2Ae^{-t} = 2e^{-t} \implies \\ A = -1. \end{gathered}$$

Then

$$y_p(t) = -te^{-t}$$

and the final solution is

$$y(t) = c_1e^t + c_2e^{-t} - te^{-t}.$$

The Method

To find a particular solution to the DE

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, use the form

$$(\#) \quad y_p(t) = t^s(A_m t^m + A_{m-1}t^{m-1} + \cdots + A_1 t + A_0)e^{rt},$$

with

(1) $s = 0$ if r is not a root of the associated auxiliary equation;

(2) $s = 1$ if r is a simple root of the associated auxiliary equation; and

(3) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the DE

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos(\beta t) \\ \quad\text{or} \\ Ct^m e^{\alpha t}\sin(\beta t) \end{cases}$$

for $\beta \neq 0$, use the form

$$\begin{aligned}(\#\#) \quad y_p(t) &= t^s(A_m t^m + A_{m-1}t^{m-1} + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) \\ &\quad + t^s(B_m t^m + B_{m-1}t^{m-1} + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t),\end{aligned}$$

with

(4) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and

(5) $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.

NOTE. The nonhomogeneity Ct^m corresponds to the case where $r = 0$.

PROBLEM (Page 182 # 26). Solve $y'' + 2y' + 2y = 4te^{-t}\cos t$.

SOLUTION. The auxiliary equation is $r^2 + 2r + 2 \Longrightarrow$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \Longrightarrow$$

$$y_h(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

For the nonhomogeneous part, we need to use (##). With $\alpha = -1$ and $\beta = 1$, $\alpha + i\beta$ is a solution to the auxiliary equation, so from (5), $s = 1$ and

$$y_p(t) = t\big[(A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t\big]e^{-t} = \big[(A_1t^2 + A_0t)\cos t + (B_1t^2 + B_0t)\sin t\big]e^{-t},$$

$$y_p'(t) = -\Big[\big((A_1 - B_1)t^2 + (A_0 - B_0 - 2A_1)t - A_0\big)\cos t + \big((A_1 + B_1)t^2 + (A_0 + B_0 - 2B_1)t - B_0\big)\sin t\Big]e^{-t},$$

$$y_p''(t) = 2\Big[\big(-B_1t^2 + (-B_0 - 2A_1 + 2B_1)t - A_0 + B_0 + A_1\big)\cos t + \big(A_1t^2 + (A_0 - 2A_1 - 2B_1)t - A_0 - B_0 + B_1\big)\Big]e^{-t}.$$

Then

$$(4B_1t + 2B_0 + 2A_1)e^{-t}\cos t + (-4A_1t - 2A_0 + 2B_1)e^{-t}\sin t = 4te^{-t}\cos t,$$

so

$$4B_1 = 4 \Longrightarrow B_1 = 1, \quad B_0 + A_1 = 0, \quad A_1 = 0 \Longrightarrow B_0 = 0,$$
$$-A_0 + B_1 = 0 \Longrightarrow A_0 = 1.$$

Thus

$$y_p(t) = (t\cos t + t^2\sin t)e^{-t} \Longrightarrow$$
$$y(t) = y_h(t) + y_p(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t + (t\cos t + t^2\sin t)e^{-t}.$$

□

PROBLEM (Page 182 # 22). Solve $x''(t) - 2x'(t) + x(t) = 24t^2e^t$.

SOLUTION. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 \Longrightarrow r = 1, 1$. Thus $x_h(t) = c_1e^t + c_2te^t$.

For the nonhomogeneous part, we need to use (#). Since our auxiliary equation has a double root, $s = 2$ by (3),giving us

$$x_p(t) = t^2(A_2t^2 + A_1t + A_0)e^t = (A_2t^4 + A_1t^3 + A_0t^2)e^t.$$

Then

$$x_p'(t) = \left[A_2t^4 + (4A_2 + A_1)t^3 + (3A_1 + A_0)t^2 + 2A_0t\right]e^t$$

and

$$x_p''(t) = \left[A_2t^4 + (8A_2 + A_1)t^3 + (12A_2 + 6_1 + A_0)t^2 + (6A_1 + 4A_0)t + 2A_0\right]e^t,$$

yielding the equation

$$L[x(t)] = x_p'' - 2x_p' + x_p = [12A_2t^2 + 6A_1t + 2A_0]e^t = 24t^2e^t.$$

Equating coefficients, we see that

$$A_2 = 2 \text{ and } A_1 = A_0 = 0.$$

Thus

$$x_p(t) = 2t^4e^t,$$

giving a final solution of

$$x(t) = x_h(t) + x_p(t) = c_1e^t + c_2te^t + 2t^4e^t.$$

□

Problem (Page 182 # 36). Find a particular solution to

$$y^{(4)} - 3y'' - 8y = \sin t.$$

Solution.

We use the form (##) with $\alpha = 0$ and $\beta = 1$. We note that i is not a root of the auxiliary equation $r^4 - 3r^2 - 8 = 0$, so we choose $s = 0$. Then

$$\begin{aligned} y_p(t) &= A_0 \cos t + B_0 \sin t \Longrightarrow \\ y_p'(t) &= -A_0 \sin t + B_0 \cos t \Longrightarrow \\ y_p''(t) &= -A_0 \cos t - B_0 \sin t \Longrightarrow \\ y_p'''(t) &= A_0 \sin t - B_0 \cos t \Longrightarrow \\ y_p^{(4)}(t) &= A_0 \cos t + B_0 \sin t. \end{aligned}$$

Then

$$y_p^{(4)} - 3y_p'' - 8y_p = -4A_0 \cos t - 4B_0 \sin t = \sin t \Longrightarrow$$

$$A_0 = 0 \text{ and } B_0 = -\frac{1}{4} \Longrightarrow$$

$$y_p(t) = -\frac{\sin t}{4}.$$

□

5. The Superposition Principle and Undetermined Coefficients Revisited

THEOREM (3 — Superposition Principal). *Let $y_1(t)$ be a solution to*

$$ay'' + by' + cy = f_1(t)$$

and $y_2(t)$ be a solution to

$$ay'' + by' + cy = f_2(t).$$

Then the linear combination

$$k_1y_1(t) + k_2y_2(t),$$

is a solution of

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)$$

regardless of the values of the constants k_1 and k_2.

PROOF. Let $y_1(t)$ be a solution to $ay'' + by' + cy = f_1(t)$ and $y_2(t)$ be a solution to $ay''+by'+cy = f_2(t)$ where $L = aD^2+bD+c$. Then $L[y_1(t)] = f_1$ and $L[y_2(t)] = f_2$. Thus for all constants k_1 and k_2,

$$L[k_1y_1(t) + k_2y_2(t)] = k_1L[y_1(t)] + k_2L[y_2(t)] = k_1f_1(t) + k_2f_2(t),$$

so $k_1y_1(t) + k_2y_2(t)$ is a solution to

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t).$$

□

EXAMPLE. Solve $y'' - y = 2e^{-t} - 4te^{-t} + 10\cos 2t$.

From a previous Example (1),

$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

For the particular solution, we will use superposition. For

$$L[y] = (2 - 4t)e^{-t},$$

we use (#) with $s = 1$ since -1 is a simple root of $r^2 - 1 = 0$. We thus choose

$$y_p(t) = t(A + Bt)e^{-t} = (At + Bt^2)e^{-t} \implies$$
$$y_p'(t) = [A + (2B - A)t - Bt^2]e^{-t} \implies$$
$$y_p''(t) = [(2B - 2A) + (A - 4B)t + Bt^2]e^{-t} \implies$$
$$L[y_p] = (2B - 2A)(e^{-t}) - 4Bte^{-t} = 2e^{-t} - 4te^{-t} \implies$$
$$2B - 2A = 2 \text{ and } -4B = -4 \implies$$
$$B = 1 \implies A = 0.$$

Then

$$y_p(t) = t^2 e^{-t}.$$

For

$$L[y] = 10\cos 2t,$$

we use (##) with $m = 1$, $\alpha = 0$, $\beta = 2$, and $s = 0$ since i is not a root of $r^2 - 1 = 0$, i.e,

$$y_q(t) = A\cos 2t + B\sin 2t \implies$$
$$y_q'' = -4A\cos 2t - 4B\sin 2t \implies$$

$$L[y_q] = (-4A\cos 2t - 4B\sin 2A) - (A\cos 2t + B\sin 2t) =$$
$$-5A\cos 2t - 5B\sin 2t = 10\cos 2t \implies$$
$$A = -2 \text{ and } B = 0.$$

Then

$$y_q(t) = -2\cos 2t \implies$$
$$y(t) = y_h(t) + y_p(t) + y_q(t) = c_1 e^t + c_2 e^{-t} + t^2 e^{-t} - 2\cos 2t.$$

The next theorem addresses IVP's:

THEOREM (4 — Existence and Uniqueness: Nonhomogeneous Case). *For any ral numbers $a(\neq 0)$, b, c, t_0,Y_0, and Y_1, suppose $y_p(t)$ is a particular solution to*

$$(*) \quad ay'' + by' + cy = f(t)$$

in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are l.i. solutions to the associated homogeneous equation in I. Then there exists a unique solution in I to the IVP

$$ay'' + by' + cy = f(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1,$$

and it is given by

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

for the appropriate choice of constants c_1 and c_2

PROBLEM (Page 187 # 2c). Given that $y_1(t) = \cos t$ is a solution to

$$y'' - y' + y = \sin t$$

and $y_2(t) = \dfrac{e^{2t}}{3}$ is a solution to

$$y'' - y' + y = e^{2t},$$

use the superposition principle to find a solution to

$$y'' - y' + y = 4 \sin t + 18e^{2t}.$$

SOLUTION. Let $g_1(t) = \sin t$ and $g_2(t) = e^{2t}$. Since

$$4 \sin t + 18e^{2t} = 4g_1(t) + 18g_2(t),$$

the function

$$y(t) = 4y_1(t) + 18y_2(t) = 4 \cos t + 6e^{2t}$$

is a solution to

$$y'' - y' + y = 4 \sin t + 18e^{2t}.$$

□

Method of Undetermined Coefficients (Revisited)

To find a particular solution to the DE

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$(\#) \quad y_p(t) = t^s(A_m t^m + A_{m-1}t^{m-1} + \cdots + A_1 t + A_0)e^{rt},$$

with

(1) $s = 0$ if r is not a root of the associated auxiliary equation;

(2) $s = 1$ if r is a simple root of the associated auxiliary equation; and

(3) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the DE

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos(\beta t) + Q_n(t)e^{\alpha t}\sin(\beta t), \quad \beta \neq 0,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$\begin{aligned}(\#\#) \quad y_p(t) = {} & t^s(A_k t^k + A_{k-1}t^{k-1} + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) \\ & + t^s(B_k t^k + B_{k-1}t^{k-1} + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t),\end{aligned}$$

where k is the larger of m and n, with

(4) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and

(5) $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.

PROBLEM (Page 188 # 22). Find a general solutiopn to

$$y''(x) + 6y'(x) + 10y(x) = 10x^4 + 24x^3 + 2x^2 - 12x + 18.$$

SOLUTION.

The auxiliary equation $r^2 + 6r + 10$ has roots

$$r = \frac{-6 \pm \sqrt{36 - 40}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i.$$

Thus

$$y_h(x) = c_1 e^{-3x} \cos x + c_2 e^{-3x} \sin x.$$

For a partcular solution, using (#) with $r = s = 0$, we choose

$$y_p(x) = A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 \Longrightarrow$$
$$y_p'(x) = 4A_4x^3 + 3A_3x^2 + 2A_2x + A_1 \Longrightarrow$$
$$y_p''(x) = 12A_4x^2 + 6A_3x + 2A_2 \Longrightarrow$$

$$\begin{aligned} y_p'' + 6y_p' + 10y_p = 10A_4x^4 + (10A_3 + 24A_4)x^3 + (10A_1 + 12A_2 + 6A_3)x^2 \\ + (10A_1 + 12A_2 + 6A_3)x + (10A_0 + 6A_1 + 2a_2) \\ = 10x^4 + 24x^3 + 2x^2 - 12x + 18 \Longrightarrow \end{aligned}$$

$$A_4 = 1, \quad A_3 = 0, \quad A_2 = -1, \quad A_1 = 0, \quad A_0 = 2.$$

Then

$$y_p(x) = x^4 - x^2 + 2,$$

so

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-3x} \cos x + c_2 e^{-3x} \sin x + x^4 - x^2 + 2.$$

□

PROBLEM (Page 188 # 30). Solve the IVP

$$y'' + 2y' + y = t^2 + 1 - e^t, \quad y(0) = 0, \quad y'(0) = 2.$$

SOLUTION.

The auxiliary equation $r^2 + 2r + 1 = 0$ has $r = 1$ as a double root. Thus

$$y_h(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

From the superposition principle, we choose

$$y_p(t) = A_2 t^2 + A_1 t + A_0 + B_0 e^t \Longrightarrow$$

$$y_p'(t) = 2A_2 t + A_1 + B_0 e^t \Longrightarrow$$

$$y_p''(t) = 2a_2 + B_0 e^t \Longrightarrow$$

$$L[y_p(t)] = A_2 t^2 + (A_1 + 4A_2)t + (A_0 + 2A_1 + 2A_2) + 4B_0 e^t = t^2 + 1 - e^t \Longrightarrow$$

$$A_2 = 1, \quad A_1 = -4A_2 = -4, \quad A_0 = 1 - aA_1 - 2A_2 = 7. \quad B_0 = -\frac{1}{4}.$$

Thus

$$y_p(t) = t^2 - 4t + 7 - \frac{e^t}{4} \Longrightarrow$$

the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 7 - \frac{e^t}{4} \Longrightarrow$$

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 2t - 4 - \frac{e^t}{4}.$$

For the initial conditions,

$$y(0) = c_1 + 7 - \frac{1}{4} = 0 \Longrightarrow c_1 = -\frac{27}{4} \text{ and}$$

$$y'(0) = -c_1 + c_2 - 4 - \frac{1}{4} = 2 \Longrightarrow c_2 = -\frac{1}{2}.$$

Thus the solution to the IVP is

$$y(t) = y_h(t) + y_p(t) = -\frac{27}{4} e^{-t} + -\frac{1}{2} t e^{-t} + t^2 - 4t + 7 - \frac{e^t}{4}.$$

$\square$

6. Variation of Parameters

This is a method for finding a particular solution for any second-order linear nonhomogeneous DE

$$(*) \quad a(t)y'' + b(t)y' + c(t)y = f(t)$$

where a, b, c, and f are continuous functions on an interval I, $a(t) \neq 0$ on I, and where we are able to find a general solution to the associated homogeneous equation.

Example (First Order). Solve $y' + 4ty = 6e^{-2t^2}$.

The associated homogeneous equation is

$$y' + 4ty = 0.$$

$$\mu = e^{\int 4t\,dt} = e^{2t^2} \Longrightarrow$$

$$e^{2t^2}y' + 4te^{2t^2}y = 0 \Longrightarrow \frac{d}{dt}\left[e^{2t^2}y\right] = 0 \Longrightarrow$$

$$e^{2t^2}y = k \Longrightarrow y_h(t) = ke^{-2t^2}$$

is a general solution to the homogeneous equation, with k as a parameter. Thus we look for a particular solution of the form

$$y_p(t) = v(t)e^{-2t^2}$$

by varying the parameter. Substituting into our DE,

$$y_p' + 4ty_p = v'(t)e^{-2t^2} + v(t)(-4te^{-2t^2}) + 4tv(t)e^{-2t^2} = v'(t)e^{-2t^2}.$$

Then, by matching up with the original equation,

$$v'(t) = 6 \Longrightarrow v(t) = 6t.$$

Then

$$y_p(t) = 6te^{-2t^2} \Longrightarrow$$

$$y(t) = ke^{-2t^2} + 6te^{-2t^2} = (6t + k)e^{-2t^2}$$

is a general solution.

We now expand on this idea in finding particular solutions for second-order equations. From Theorem 4 of the previous section, we know that what we are looking for is out there.

Assume $y_1(t)$ and $y_2(t)$ are l.i. solutions of the homogeneous DE associated with $(*)$. Then

$$y_h(t) = c_1y_1(t) + c_2y_2(t)$$

is a general solution of the homogeneous DE. We need to find $v_1(t)$, $v_2(t)$ such that

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

is a solution of $(*)$.

$$y_p' = (v_1y_1' + v_2y_2') + (v_1'y_1 + v_2'y_2),$$

$$y_p'' = (v_1y_1'' + v_2y_2'') + (v_1'y_1' + v_2'y_2') + \frac{d}{dt}(v_1'y_1 + v_2'y_2).$$

Suppose

$$v_1'y_1 + v_2'y_2 = 0.$$

Then the LHS (left-hand-side) of $(*)$ is

$$\begin{aligned} ay_p'' &= av_1y_1'' + av_2y_2'' + a(v_1'y_1' + v_2'y_2') \\ +by_p' &= bv_1y_1' + bv_2y_2' \\ +cy_p &= cv_1y_1 + cv_2y_2 \end{aligned}$$

$$L[y_p] = v_1L[y_1] + v_2L[y_2] + a(v_1'y_1' + v_2'y_2')$$

Since y_1 and y_2 are solutions of the associated DE,

$$L[y_1] = 0 \text{ and } L[y_2] = 0 \Longrightarrow$$
$$L[y_p] = a(v_1'y_1' + v_2'y_2').$$

Then y_p will be a solution of $(*)$ if also $L[y_p] = f$, or altogether if the system

$$(\Box) \quad \begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = \dfrac{f}{a} \end{cases}$$

has a solution for v_1', v_2'.

To eliminate u_2',

$$\begin{aligned} y_1 y_2' v_1' + y_2 y_2' v_2' &= 0 \\ -y_1' y_2 v_1' - y_2 y_2' v_2' &= -\frac{f y_2}{a} \end{aligned}$$

$$v_1'(y_1 y_2' - y_1' y_2) = -\frac{f y_2}{a}.$$

Similarly, eliminating v_1',

$$v_2'(y_1 y_2' - y_1' y_2) = -\frac{f y_1}{a}.$$

Thus, the system has a solution if

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 \neq 0.$$

But since $y_1(t)$ and $y_2(t)$ are l.i., Lemma 1 from Section 2 of this chapter says that this is always the case. Then the solutions of $(\Box)$ are

$$v_1'(t) = -\frac{f(t) y_2(t)}{a(t) W(t)} \text{ and } v_2'(t) = \frac{f(t) y_1(t)}{a(t) W(t)} \Longrightarrow$$

$$v_1(t) = -\int \frac{f(t) y_2(t)}{a(t) W(t)}\, dt \text{ and } v_2(t) = \int \frac{f(t) y_1(t)}{a(t) W(t)}\, dt \Longrightarrow$$

$$y_p(t) = -y_1(t) \int \frac{f(t) y_2(t)}{a(t) W(t)}\, dt + y_2(t) \int \frac{f(t) y_1(t)}{a(t) W(t)}\, dt.$$

Method of Variation of Parameters

To find y_p, a particular solution of

$$(*) \quad a(t)y'' + b(t)y' + c(t)y = f(t) :$$

(1) Find

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t),$$

the solution of the homogeneous DE associated with $(*)$.

(2) Solve

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = \dfrac{f}{a} \end{cases}$$

for $v_1'(t)$ and $v_2'(t)$.

(3) Find $v_1(t)$ and $v_2(t)$ by integration with 0 as the constants of integration.

(4) The particular solution is then

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).$$

In solving the system of equations in (2), it is often useful to use

THEOREM (Cramer's Rule). *If* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0,$ *the solution of*

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

is

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Example.

(1) Solve $y'' + y = \tan t$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

$$r^2 + 1 = 0 \Longrightarrow r^2 = -1 \Longrightarrow r = \pm i \Longrightarrow \alpha = 0 \text{ and } \beta = 1 \Longrightarrow$$

$$y_h(t) = c_1 \cos t + c_2 \sin t \Longrightarrow y_1(t) = \cos t \text{ and } y_2(t) = \sin t.$$

This yields the system of equations

$$\begin{cases} (\cos t)v_1' + (\sin t)v_2' = 0 \\ -(\sin t)v_1' + (\cos t)v_2' = \tan t \end{cases}$$

We use Cramer's Rule to solve for v_1' and v_2'.

$$v_1' = \frac{\begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = -\sin t \tan t$$

$$v_2' = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix}}{1} = \cos t \tan t$$

$$v_1(t) = -\int \sin t \tan t \, dt = -\int \frac{\sin^2 t}{\cos t} \, dt = -\int \frac{1 - \cos^2 t}{\cos t} \, dt =$$

$$\int (\cos t - \sec t) \, dt = \sin t - \ln(\sec t + \tan t)$$

$$v_2(t) = \int \cos t \tan t \, dt = \int \sin t \, dt = -\cos t$$

Then

$$y_p(t) = [\sin t - \ln(\sec t + \tan t)] \cos t - \cos t \sin t = -\cos t \ln(\sec t + \tan t) \Longrightarrow$$

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln(\sec t + \tan t)$$

is a general solution.

(2) Solve $y'' + 4y' + 4y = te^{-2t}$.

$$r^2 + 4r + 4 = (r+2)^2 = 0 \Longrightarrow r = -2, -2 \Longrightarrow$$
$$y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t} \Longrightarrow y_1(t) = e^{-2t} \text{ and } y_2(t) = te^{-2t}.$$

This yields the system of equations

$$\begin{cases} e^{-2t}v_1' + te^{-2t}v_2' = 0 \\ -2e^{-2t}v_1' + (e^{-2t} - 2te^{-2t})v_2' = te^{-2t} \end{cases}$$

We then divide each equation by e^{-2t} to simplify.

$$\begin{cases} v_1' + tv_2' = 0 \\ -2v_1' + (1-2t)v_2' = t \end{cases}$$

We use Cramer's Rule to solve for u_1' and u_2'.

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ t & 1-2t \end{vmatrix}}{\begin{vmatrix} 1 & t \\ -2 & 1-2t \end{vmatrix}} = \frac{-t^2}{1} = -t^2$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ -2 & t \end{vmatrix}}{1} = t$$

$$v_1(t) = -\int t^2\,dt = -\frac{t^3}{3} \text{ and } v_2(t) = \int t\,dt = \frac{t^2}{2}$$

Then

$$y_p(t) = -\frac{t^3}{3}e^{-2t} + \frac{t^2}{2}te^{-2t} = \frac{1}{6}t^3 e^{-2t} \Longrightarrow$$
$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} + \frac{1}{6}t^3 e^{-2t}$$

is a general solution.

7. Variable-Coefficient Equations

We begin with an important theorem. In this theorem, we write the second order DE in what is called standard form.

Theorem (5 — Existence and Uniqueness of Solutions). *Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. Then, for any choice of initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the IVP*

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

Cauchy-Euler Equations

In general, a second-order linear homogeneous DE with non-constant coefficients

$$a(t)y'' + b(t)y' + c(t)y = 0$$

has solutions that cannot be expressed in terms of finitely many elementary functions. Such equations often require power series solutions or numerical techniques. An exception:

Definition (2 — Homogeneous Cauchy-euler Equations).

A second-order homogeneous Cauchy-Euler equation is an equation of the form

$$(*) \quad at^2y'' + bty' + cx = 0$$

where a, b, c are constants.

We will use a technique different from that of the text here.

We first seek a solution on an interval where $t > 0$. We do a change of variable

$$t = e^u \quad \text{(since } t > 0\text{)}.$$

Now

$$\frac{dt}{du} = e^u = t,$$

so

$$(\diamond) \quad \frac{dy}{du} = \frac{dy}{dt} \cdot \frac{dt}{du} = t\frac{dy}{dt}$$

and

$$\begin{aligned}\frac{d^2y}{du^2} &= \frac{d}{du}\Big(t\frac{dy}{dt}\Big)\\ &= t\frac{d}{du}\Big(\frac{dy}{dt}\Big) + \frac{dy}{dt} \cdot \frac{dt}{du}\\ &= t\frac{d^2y}{dt^2} \cdot \frac{dt}{du} + \frac{dt}{du} \cdot \frac{dy}{dt}\\ &= t^2\frac{d^2y}{dt^2} + \frac{dy}{du} \Longrightarrow\end{aligned}$$

$$(\diamond\diamond) \quad t^2\frac{d^2y}{dt^2} = \frac{d^2y}{du^2} - \frac{dy}{du}.$$

Then $(*)$ becomes

$$a\Big(\frac{d^2y}{du^2} - \frac{dy}{du}\Big) + b\frac{dy}{du} + cy = 0$$

or

$$(\#) \quad a\frac{d^2y}{du^2} + (b-a)\frac{dy}{du} + cy = 0.$$

The characteristic equation for (#) is then

$$ar^2 + (b-a)r + c = 0,$$

called the <u>Cauchy-Euler characteristic equation</u>. Since (#) has constant coefficients, we use the 3 cases discussed above to find general solutions $y(u)$. We then find general solutions $x(t)$ by using the reverse substitution

$$u = \ln t.$$

We now seek a solution on an interval where $t < 0$. In this case, we use

$$-t = e^u,$$

so

$$\frac{dy}{du} = \frac{dy}{dt} \cdot \frac{dt}{du} = \frac{dy}{dt}(-e^u) = t\frac{dy}{dt}.$$

Thus $(\diamond)$ and $(\diamond\diamond)$ remain the same as before, again yielding $(\#)$ as an equation. But now the reverse substitution is

$$u = \ln(-t).$$

Putting the two solutions together, we have the

<u>Method for Solving Homogeneous Cauchy-Euler Equations</u>

To find a general solution of

$$(*) \quad at^2y'' + bty' + cy = 0$$

on an interval that excludes $t = 0$, complete the following steps.

(1) Find a general solution of the associated equation

$$(\#) \quad a\frac{d^2y}{du^2} + (b-a)\frac{dy}{du} + cy = 0.$$

(2) After completing Step 1, replace u with $\ln|t|$ and e^u with $|t|$.

EXAMPLE.

(1) Solve $3t^2y'' + 11ty' - 3y = 0$, $t \neq 0$.

The associated characteristic equation

$$ar^2 + (b-a)r + c = 3r^2 + 8r - 3 = (3r-1)(r+3) = 0$$

has roots

$$r_1 = \frac{1}{3} \text{ and } r_2 = -3.$$

Then

$$y(u) = c_1 e^{\frac{1}{3}u} + c_2 e^{-3u} = c_1 (e^u)^{\frac{1}{3}} + c_2 (e^u)^{-3} \Longrightarrow$$
$$y(t) = c_1 |t|^{\frac{1}{3}} + c_2 |t|^{-3}.$$

(2) Solve $t^2y'' + 3ty' + 4y = 0, \quad t > 0$.

The associated characteristic equation

$$ar^2 + (b-a)r + c = r^2 + 2r + 4 = 0$$

has roots

$$r = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm \sqrt{3}i.$$

Let $\alpha = -1$ and $\beta = \sqrt{3}$. Then

$$y(u) = c_1 e^{-u} \cos(\sqrt{3}u) + c_2 e^{-u} \sin(\sqrt{3}u) \Longrightarrow$$
$$y(t) = c_1 t^{-1} \cos(\sqrt{3} \ln t) + c_2 t^{-1} \sin(\sqrt{3} \ln t).$$

(3) Solve $9t^2y'' + 15ty' + y = 0, \quad t > 0, \quad y(1) = 2, \quad y'(1) = \dfrac{7}{3}$.

The associated characteristic equation

$$ar^2 + (b-a)r + c = 9r^2 + 6r + 1 = (3r+1)^2 = 0$$

has roots

$$r_1 = r_2 = -\frac{1}{3}.$$

Then

$$y(u) = c_1 e^{-\frac{1}{3}u} + c_2 u e^{-\frac{1}{3}u} \Longrightarrow$$
$$y(t) = c_1 t^{-\frac{1}{3}} + c_2 t^{-\frac{1}{3}} \ln t = \frac{c_1 + c_2 \ln t}{t^{\frac{1}{3}}} \Longrightarrow$$

$$y'(t) = \frac{t^{\frac{1}{3}}\left(c_2 \cdot \frac{1}{t}\right) - (c_1 + c_2 \ln t)\frac{1}{3}t^{-\frac{2}{3}}}{t^{\frac{2}{3}}} = \frac{\frac{1}{3}t^{-\frac{2}{3}}[3c_2 - c_1 - c_2 \ln t]}{t^{\frac{2}{3}}} = \frac{3c_2 - c_1 - c_2 \ln t}{3t^{\frac{4}{3}}}$$

Then

$$y(1) = c_1 = 2 \text{ and}$$
$$y'(1) = \frac{3c_2 - c_1}{3} = \frac{3c_2 - 2}{3} = \frac{7}{3} \Longrightarrow 3c_2 - 2 = 7 \Longrightarrow c_2 = 3.$$

Thus the solution to the IVP is

$$y(t) = \frac{2 + 3\ln t}{t^{\frac{1}{3}}}.$$