THEOREM. There is no rational number whose square is 2.

PROOF.

We use indirect reasoning.

Suppose $x$ is a rational number whose square is 2.

Then $x$ can be written in lowest terms as $\frac{a}{b}$, where $a$ is an integer and $b$ is a positive integer.

Since $x^2 = 2$, $\left(\frac{a}{b}\right)^2 = 2$, so $\frac{a^2}{b^2} = 2$. Then $a^2 = 2b^2$, so $a^2$ is even.

But then $a$ is even, so $a = 2n$ for some integer $n$.

Then $(2n)^2 = 2b^2$, so $4n^2 = 2b^2$.

Then $2n^2 = b^2$, so $b^2$ is even, and thus $b$ is even.

Then $a$ and $b$ both have 2 as a common factor, so $\frac{a}{b}$ cannot be in lowest terms, a contradiction.

Thus $x$ cannot be rational. \qed

We have learned that every fraction can be written as a repeating decimal, and vice-versa. Then so can every rational number just by taking opposites.

Thus the irrational numbers, the numbers that are not rational, must have infinite nonrepeating decimal representations.

**Definition (The Real Numbers).** The set of real numbers, $\mathbb{R}$, is the set of all numbers that have an infinite decimal representation.