\>

Convergence of Power Series

\>

```
> restart:
```

A **power series** P(*x*) in *x* is a sum of constants times powers of *x*:

$$P(x) = C_0 + C_1\,x + C_2\,x^2 + \ldots + C_n\,x^n + \ldots = \sum_{n=0}^{\infty} C_n\,x^n.$$

An example is the Taylor series for f(*x*) = sin(*x*) about *x* = 0:

$$T(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n\,x^{(2\,n+1)}}{(2\,n+1)!} + \ldots$$

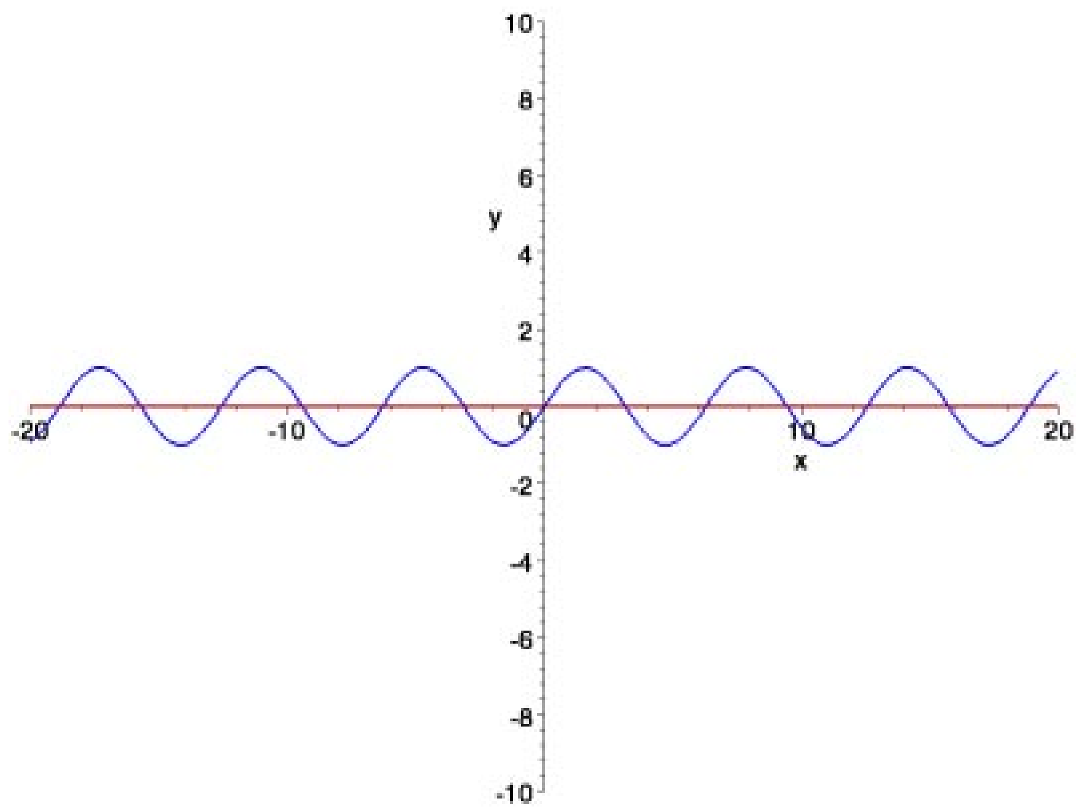
The first question of interest for a power series P(*x*) is "for what values of *x* does the power series converge?" Secondly, if the power series is a Taylor series T(*x*) for f(*x*) and converges for a given *x*, does T(*x*) = f(*x*)?

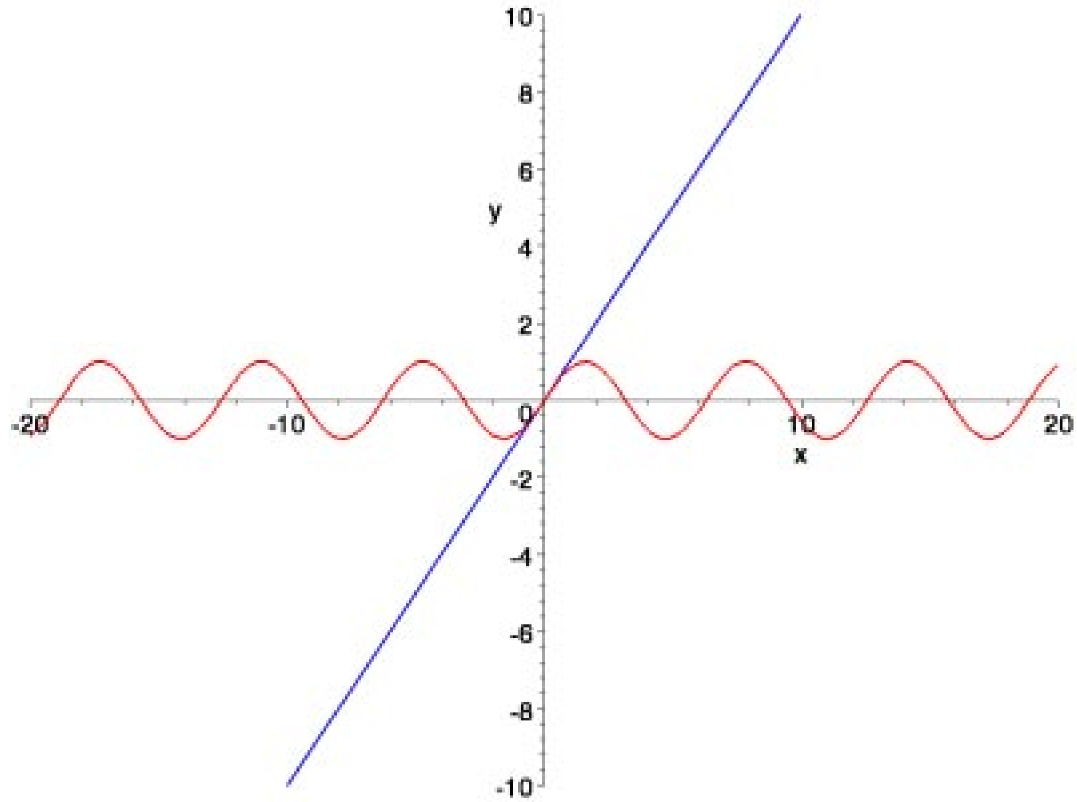
```
> f:=sin(x);
```

$$f := \sin(x)$$

Next we calculate and plot in succession the first 31 Taylor polynomials about *x* = 0.

```
> for n from 0 to 30 do
  P[n]:=convert(taylor(f,x=0,n+1),polynom):
  od:
> for n from 0 to 30 do
  plot({f,P[n]},x=-20..20,y=-10..10,color=[red,blue],thickness=3):
  od;
```

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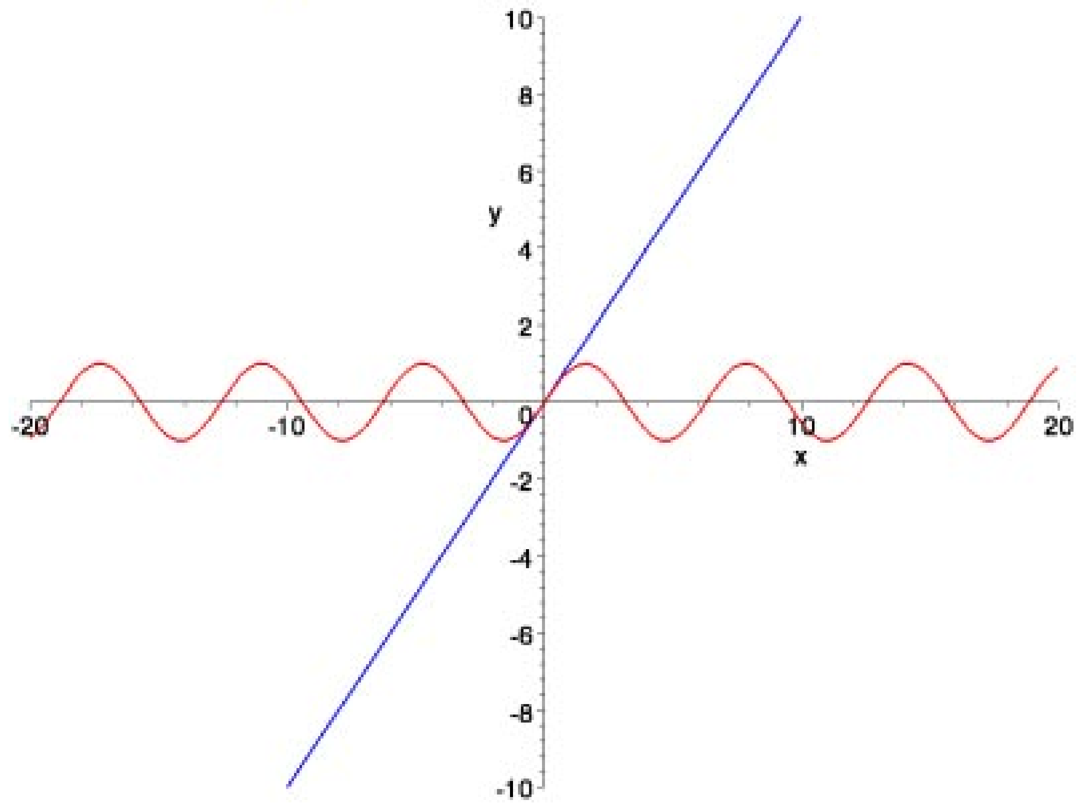
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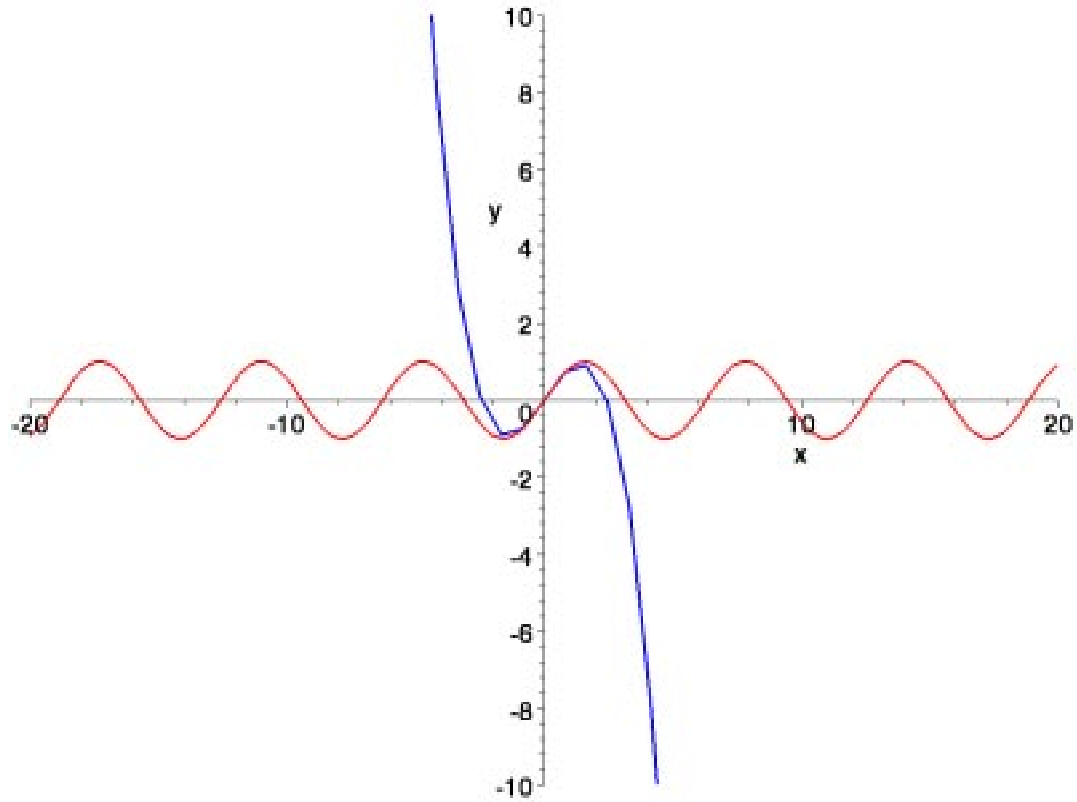

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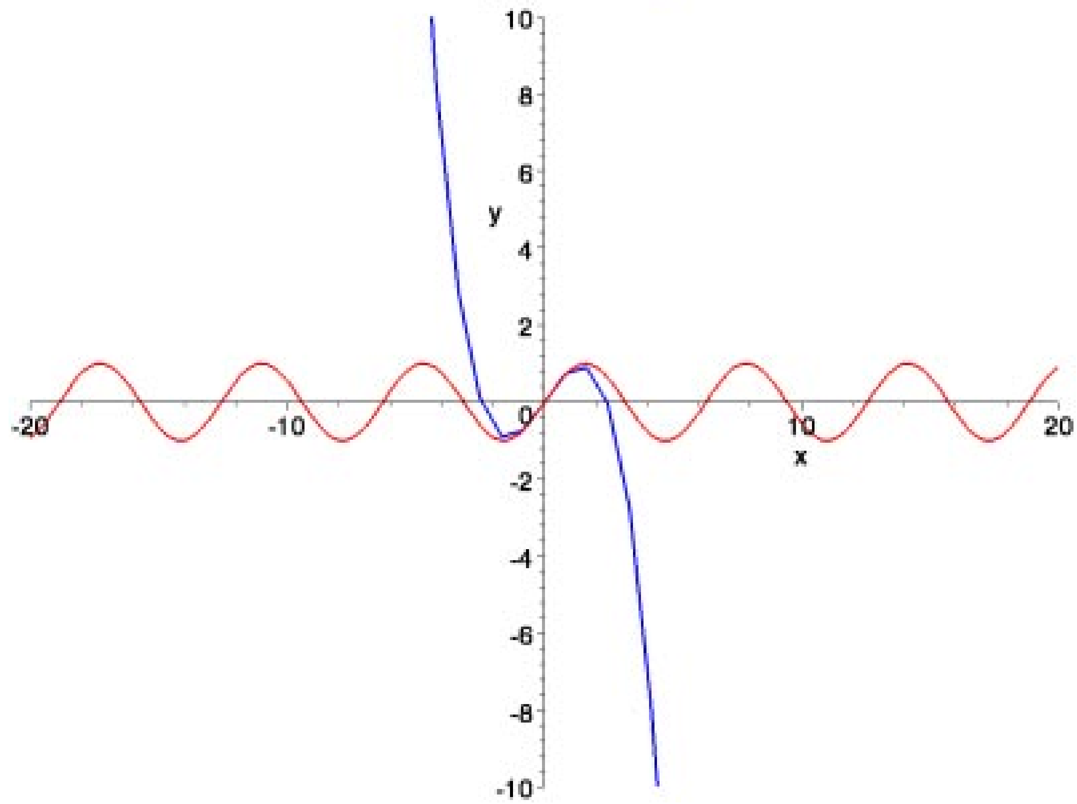

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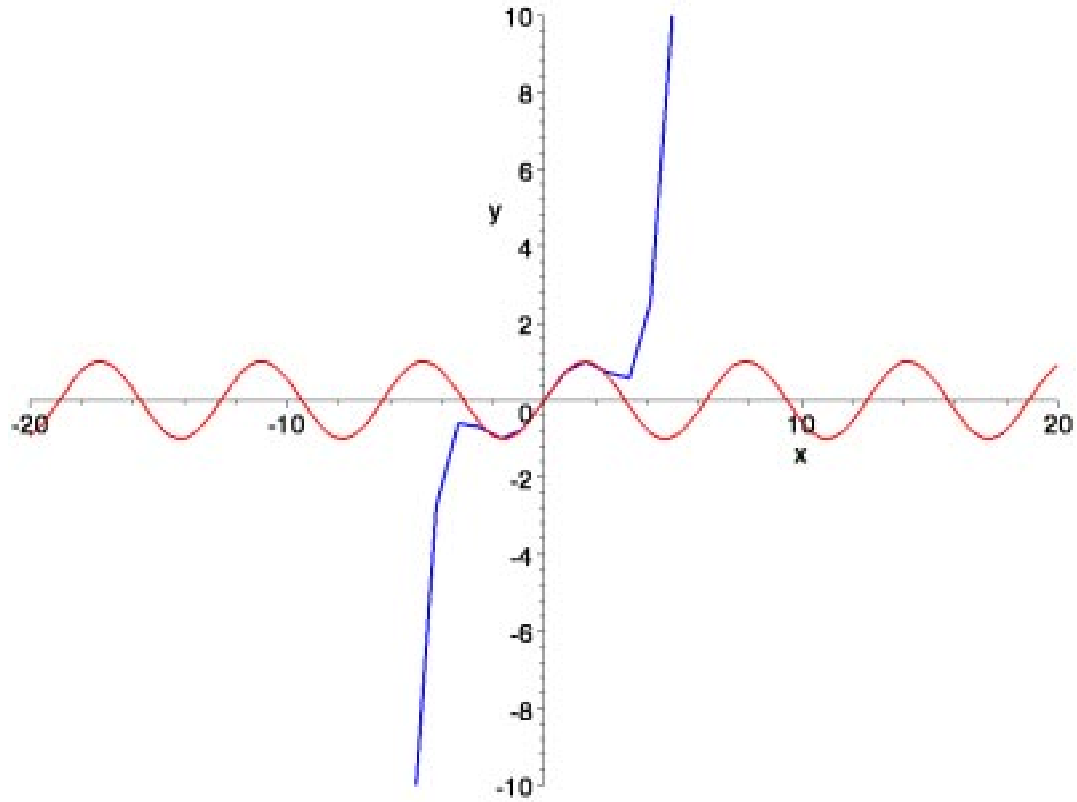
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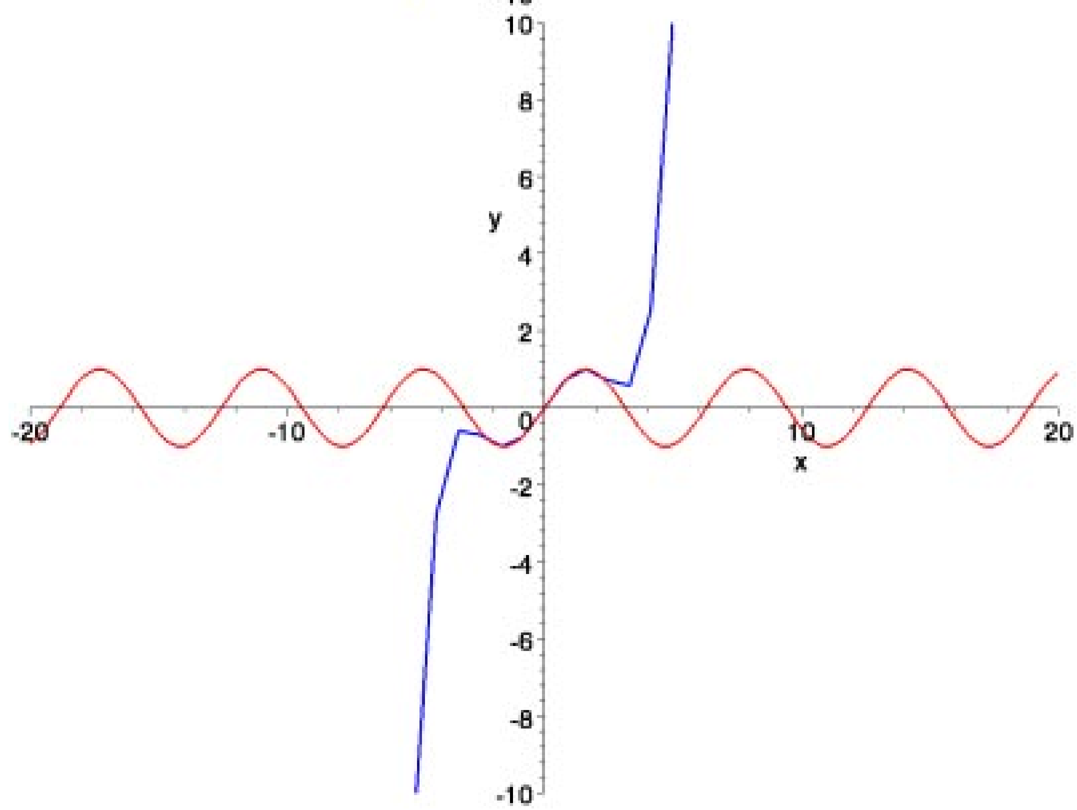
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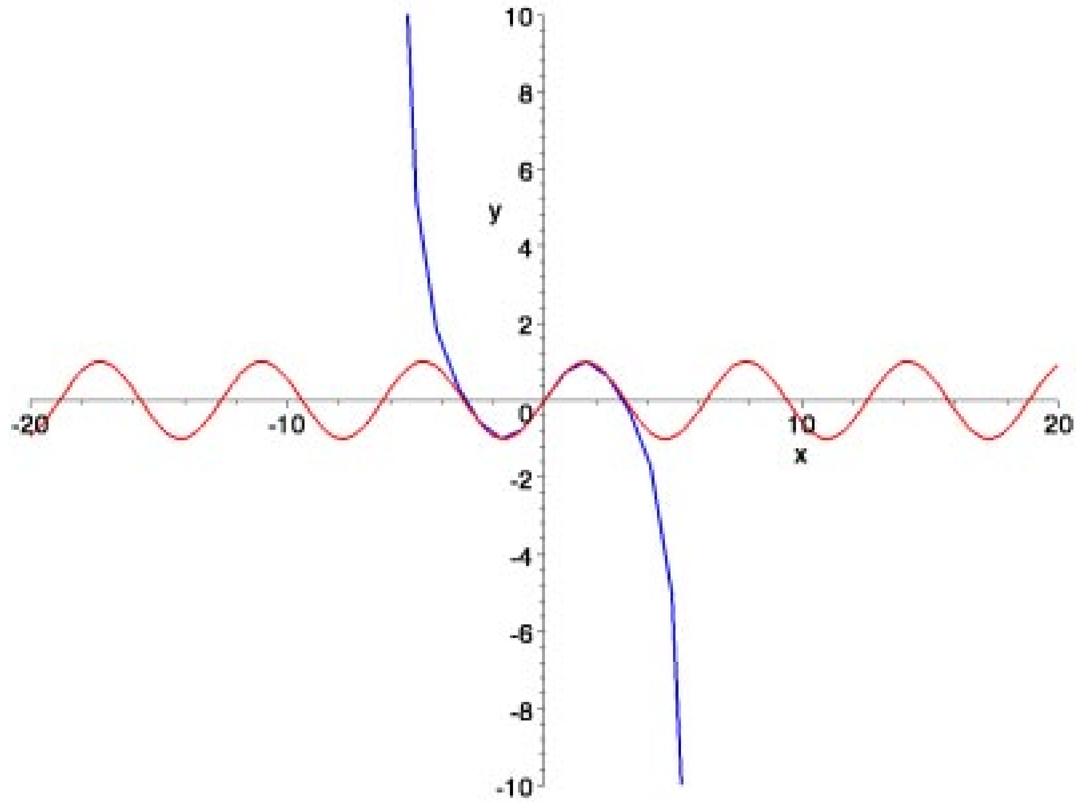
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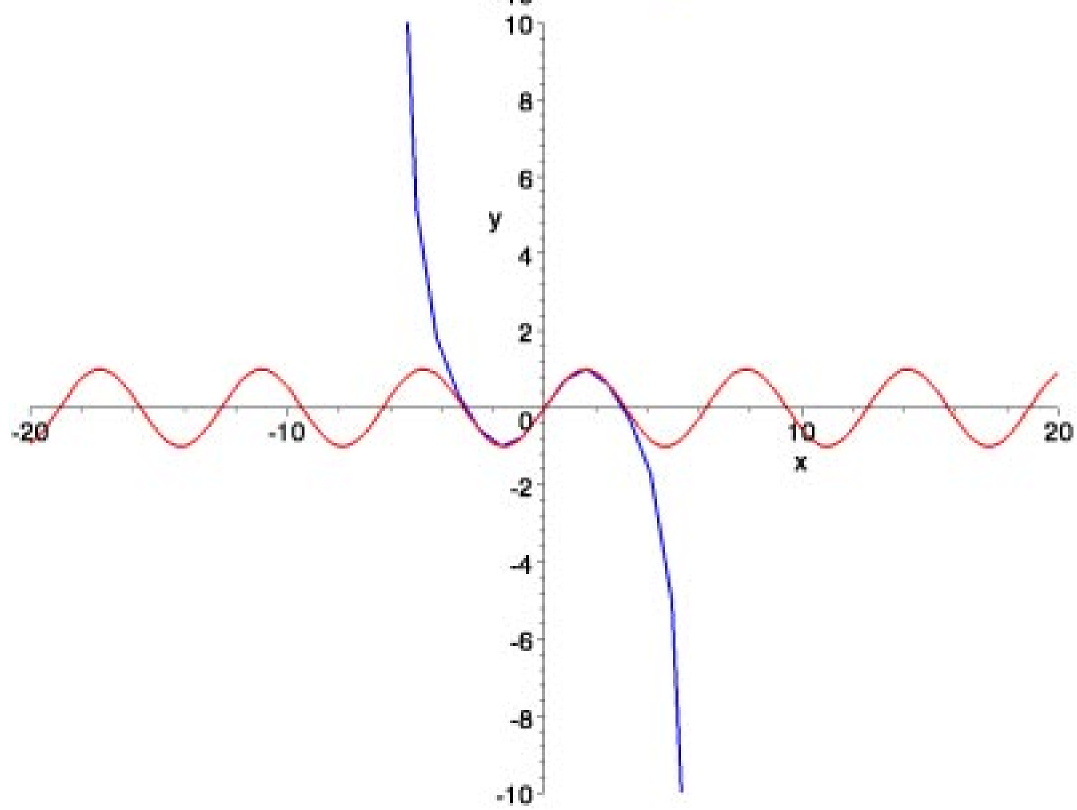
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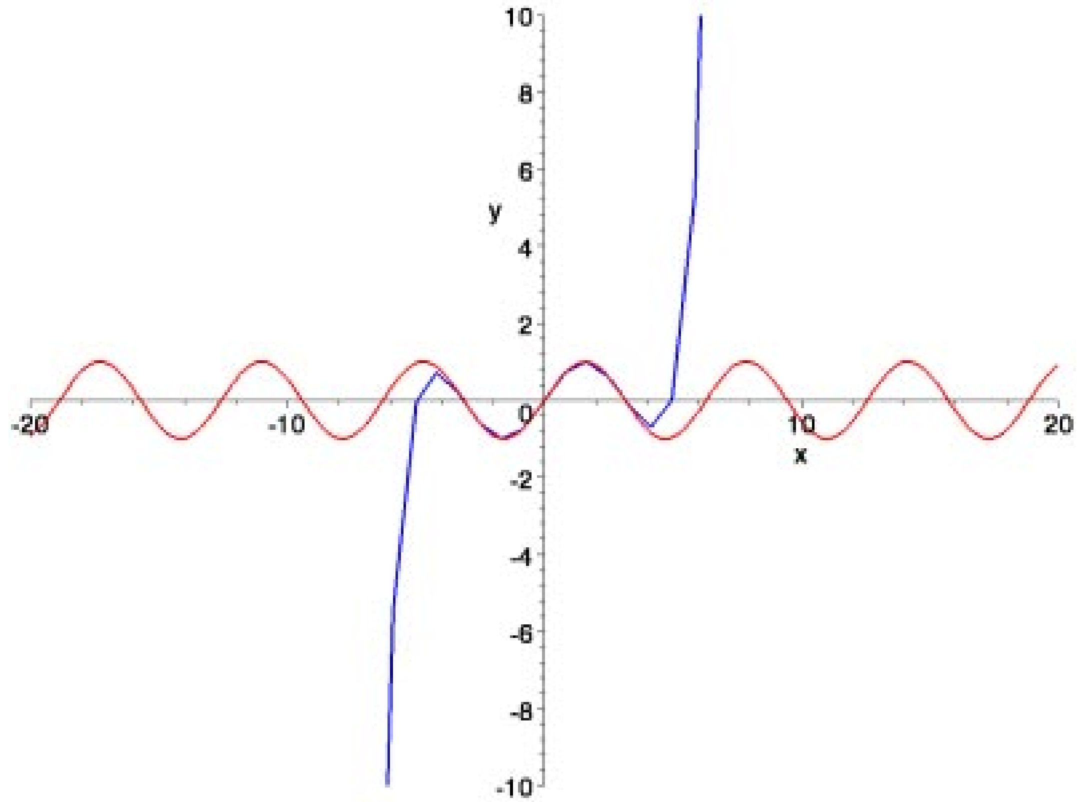
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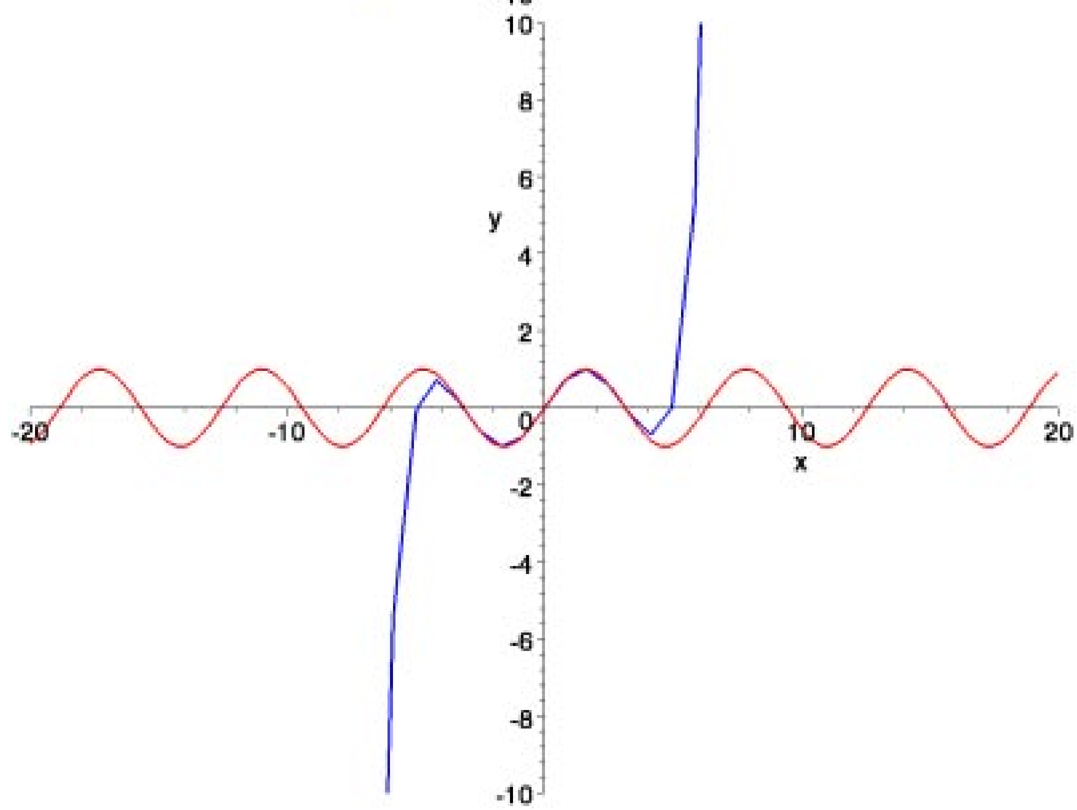
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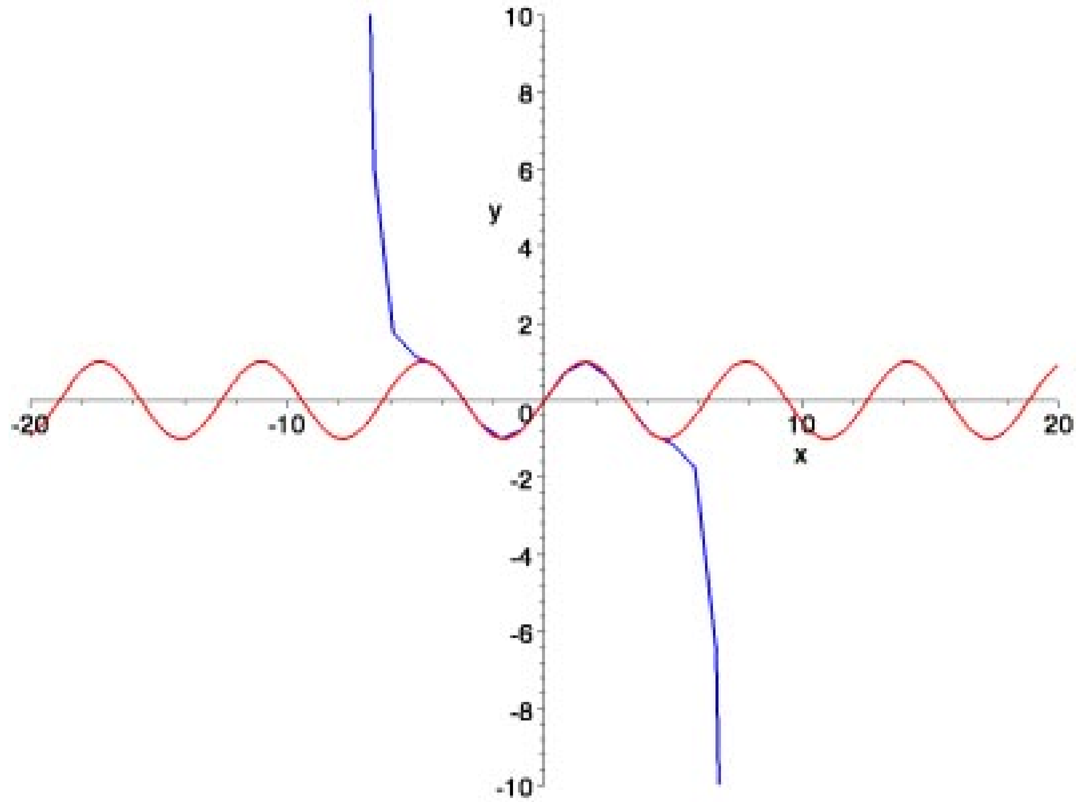

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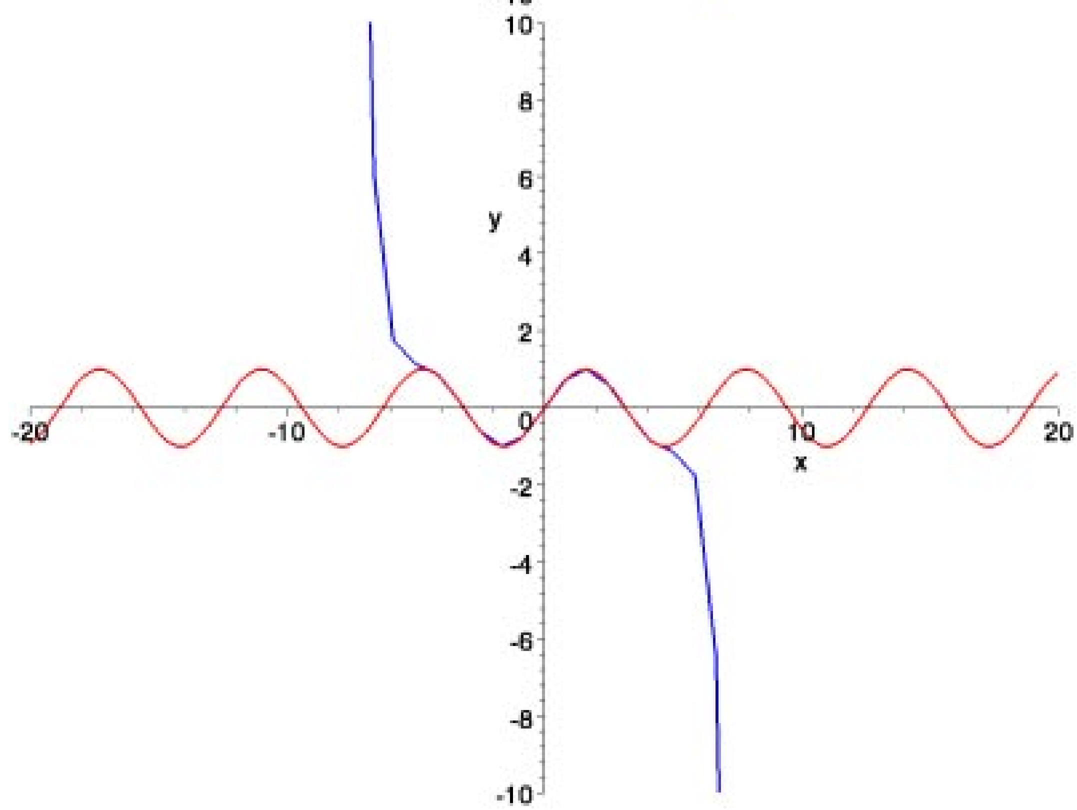

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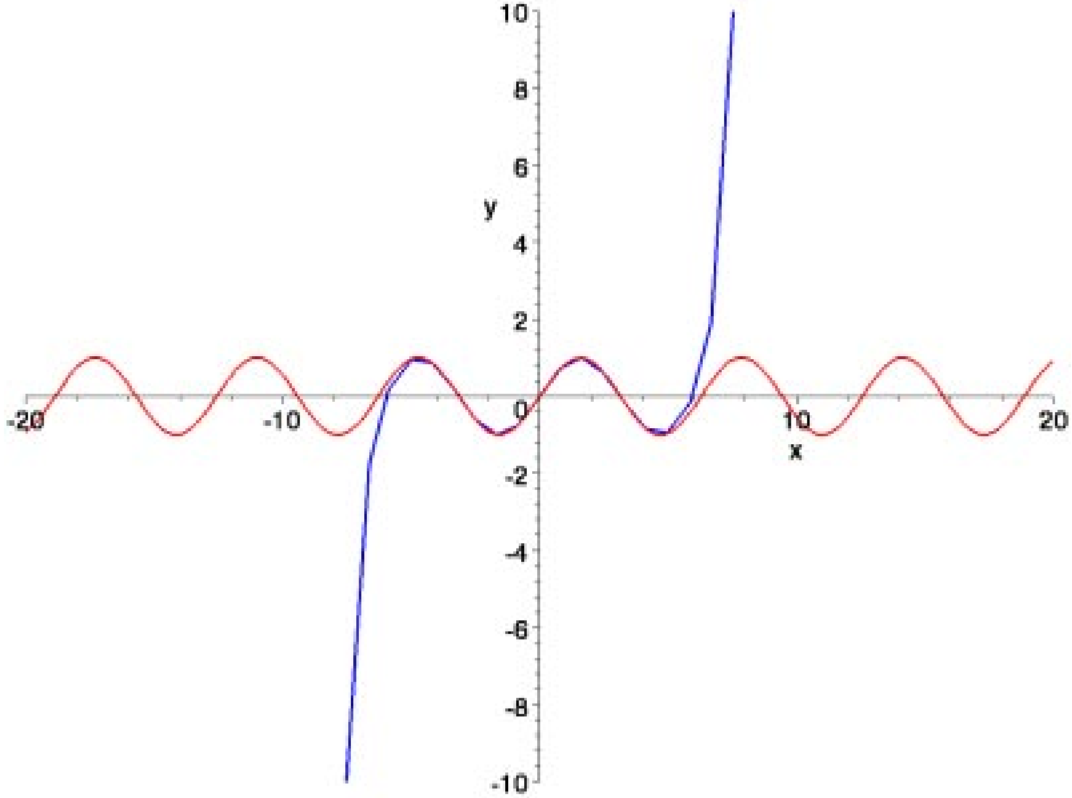

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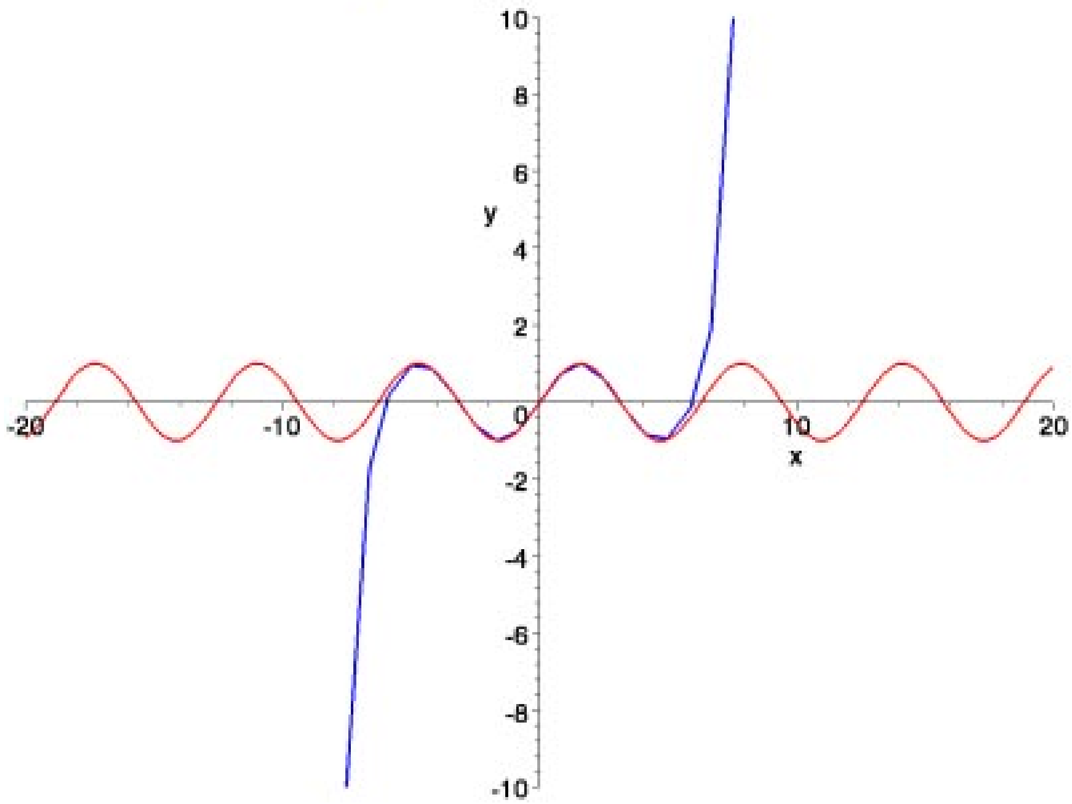

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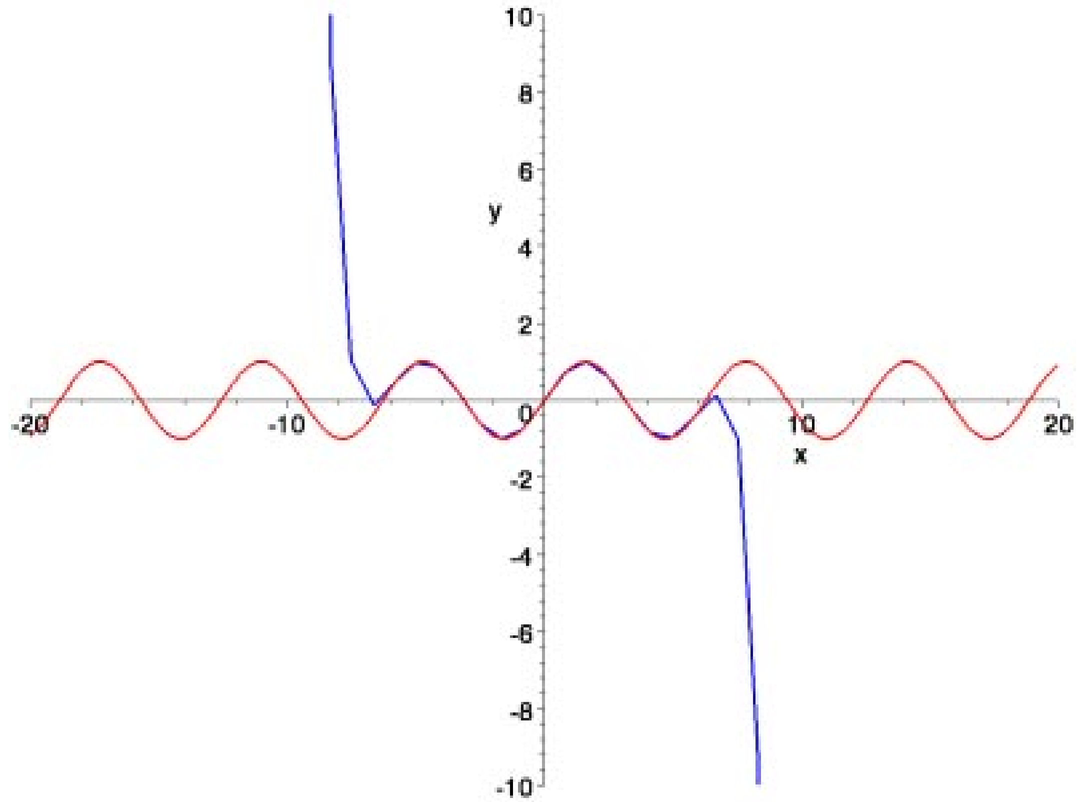

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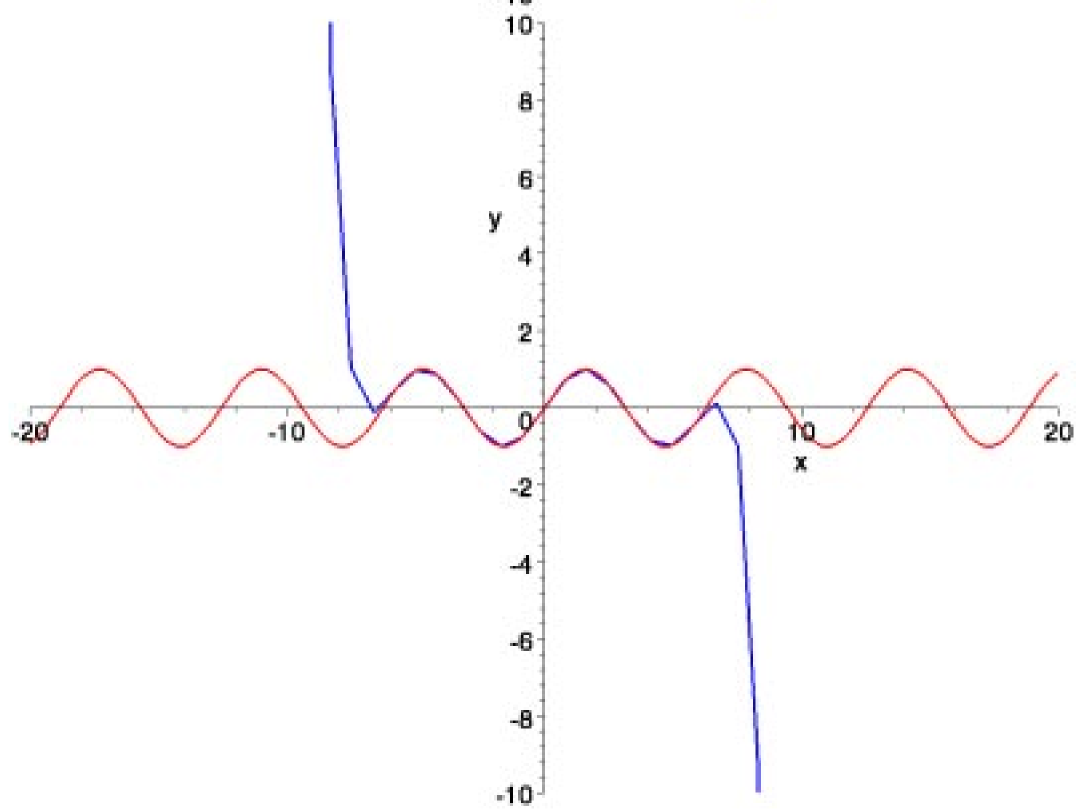

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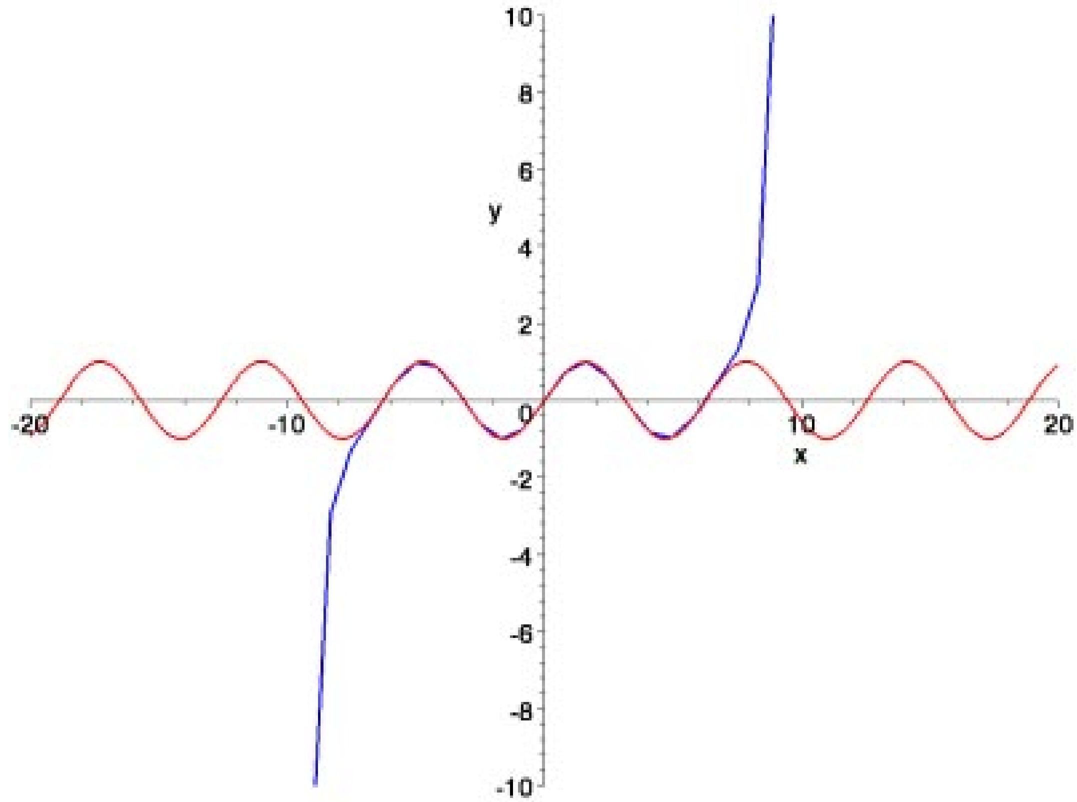

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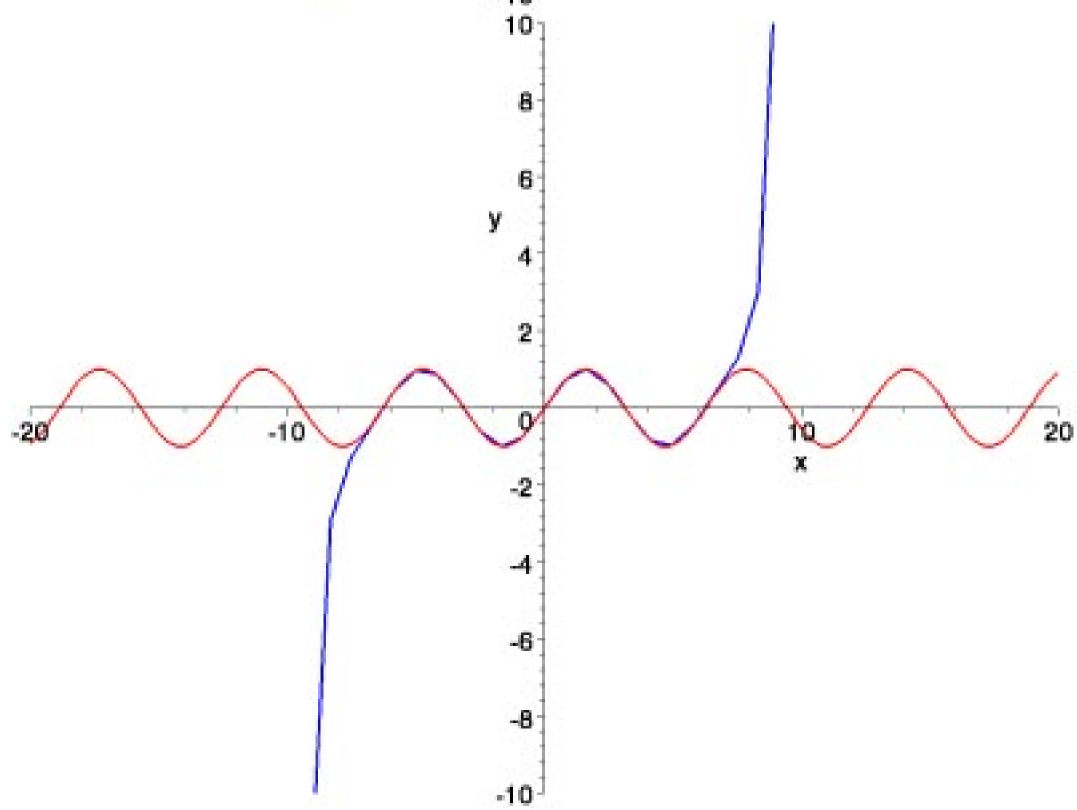

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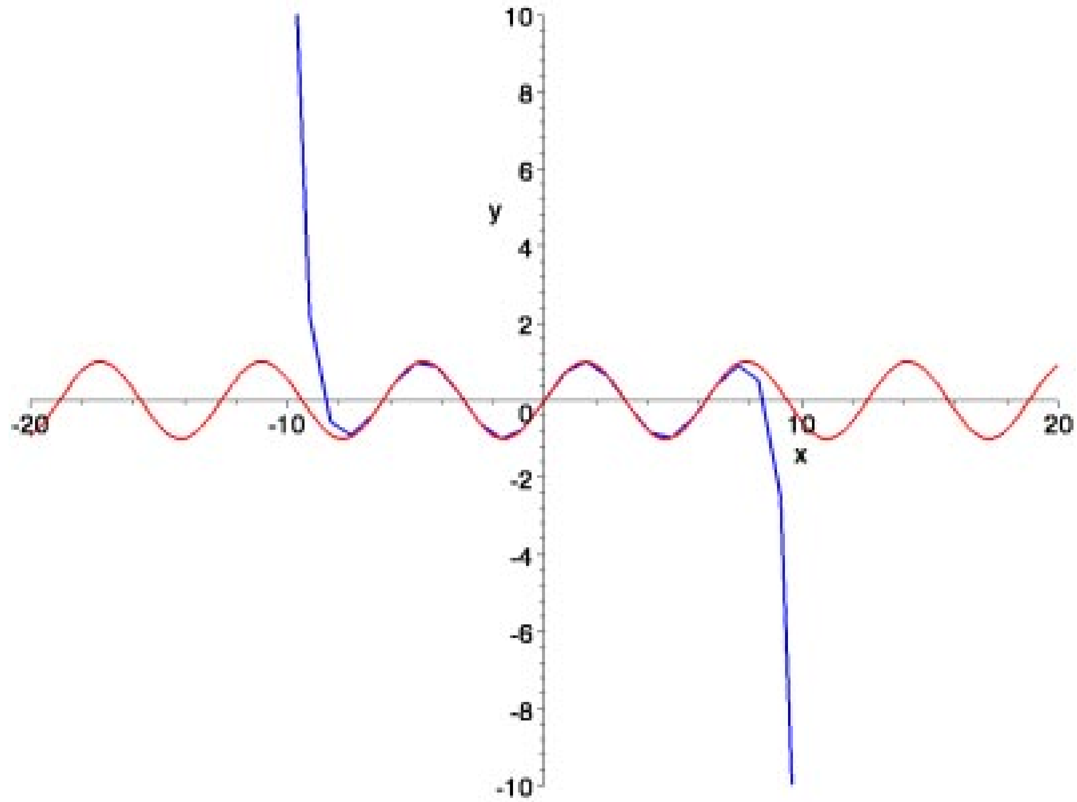
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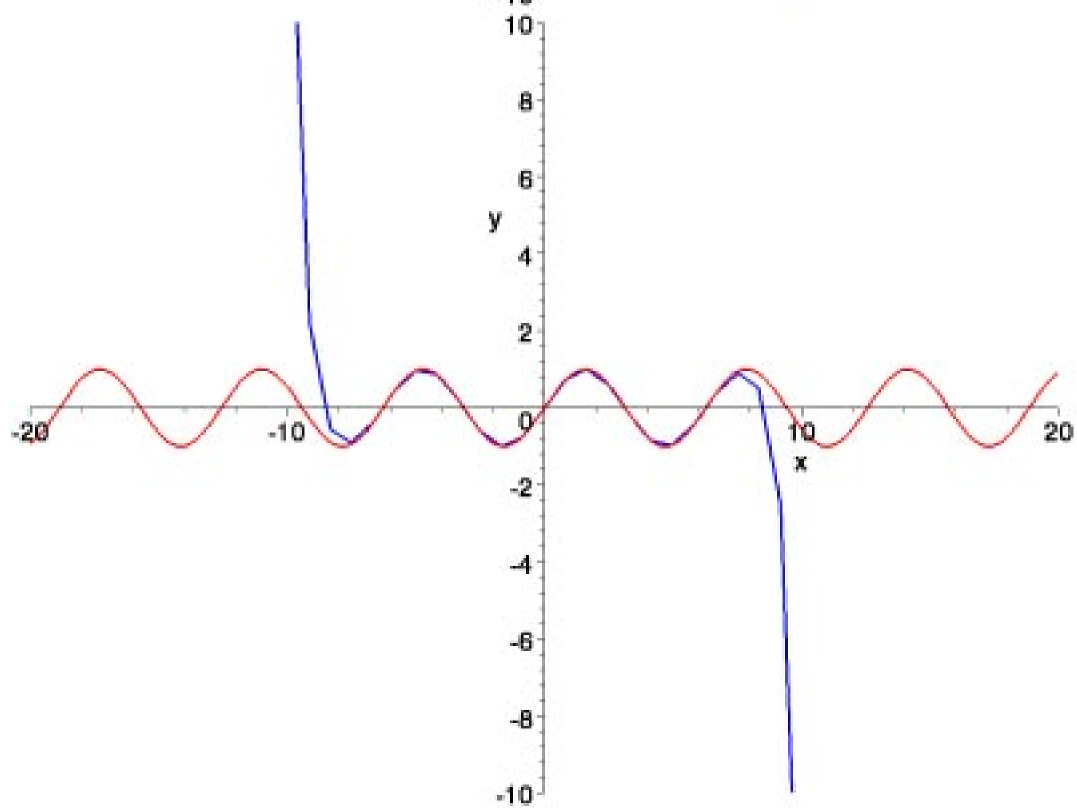
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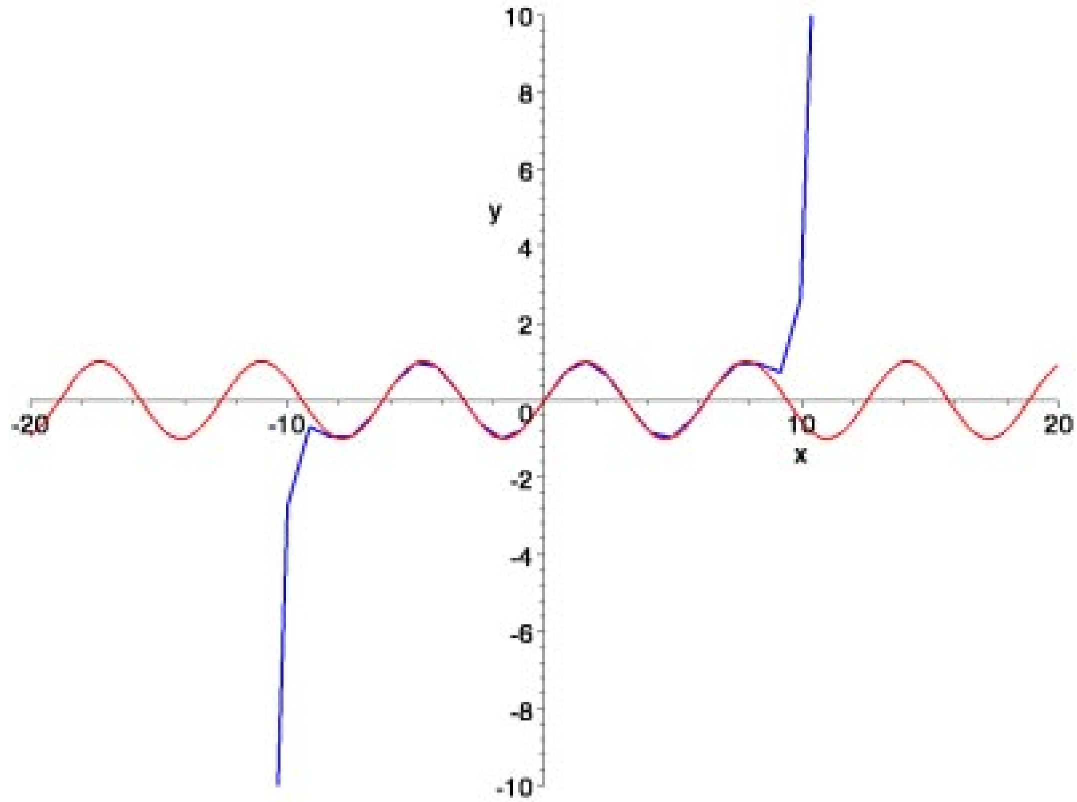
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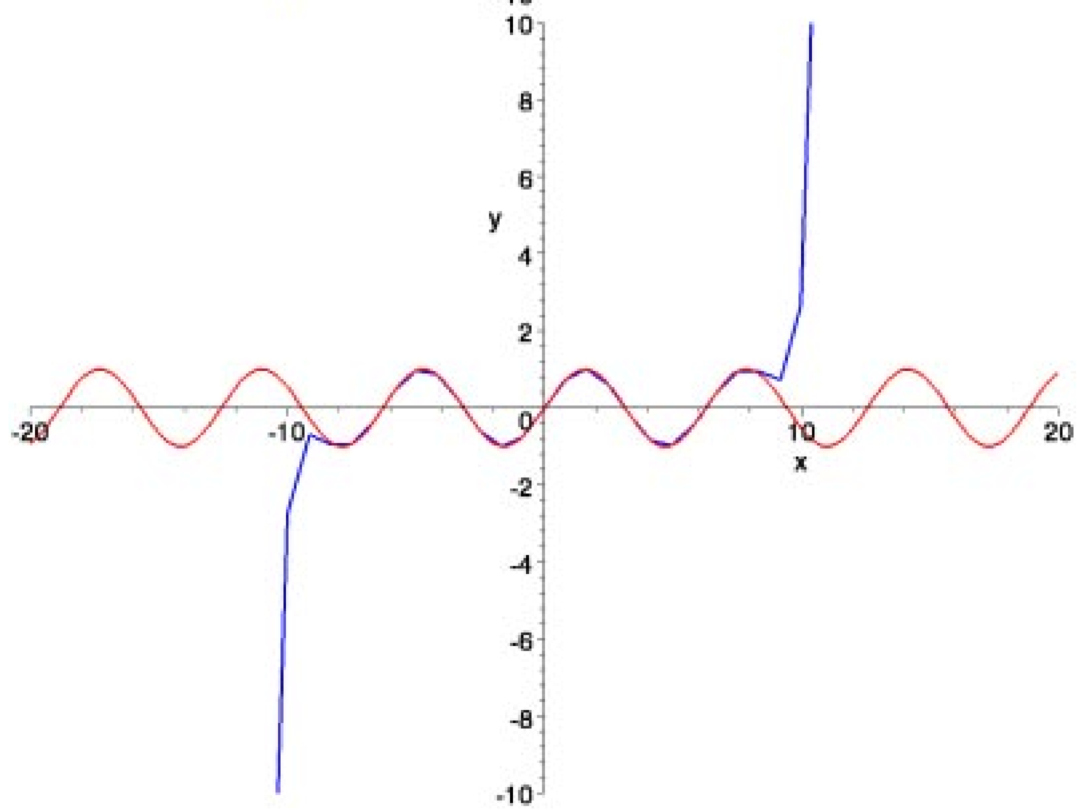
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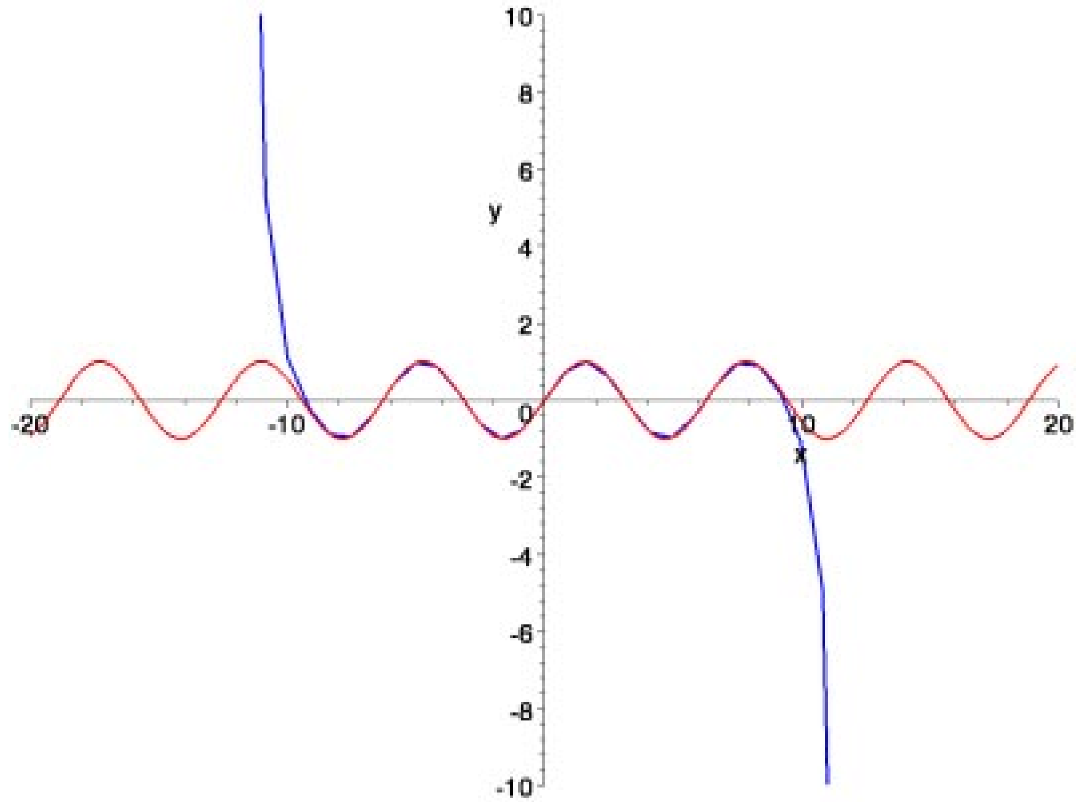
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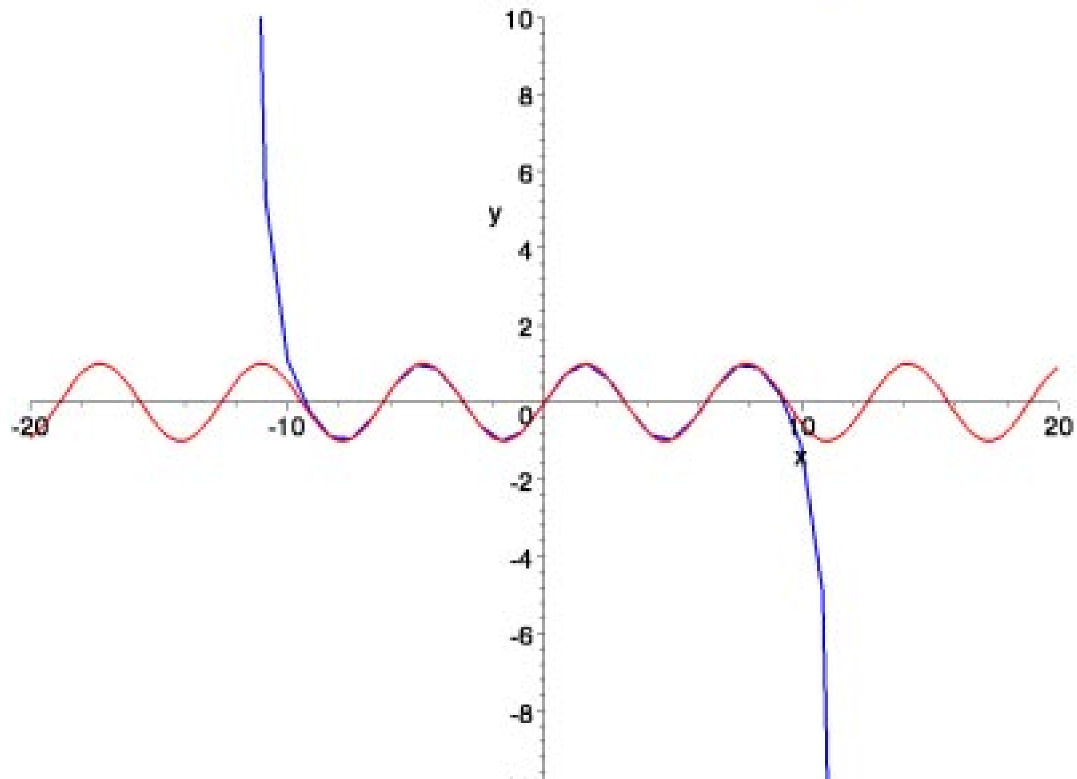
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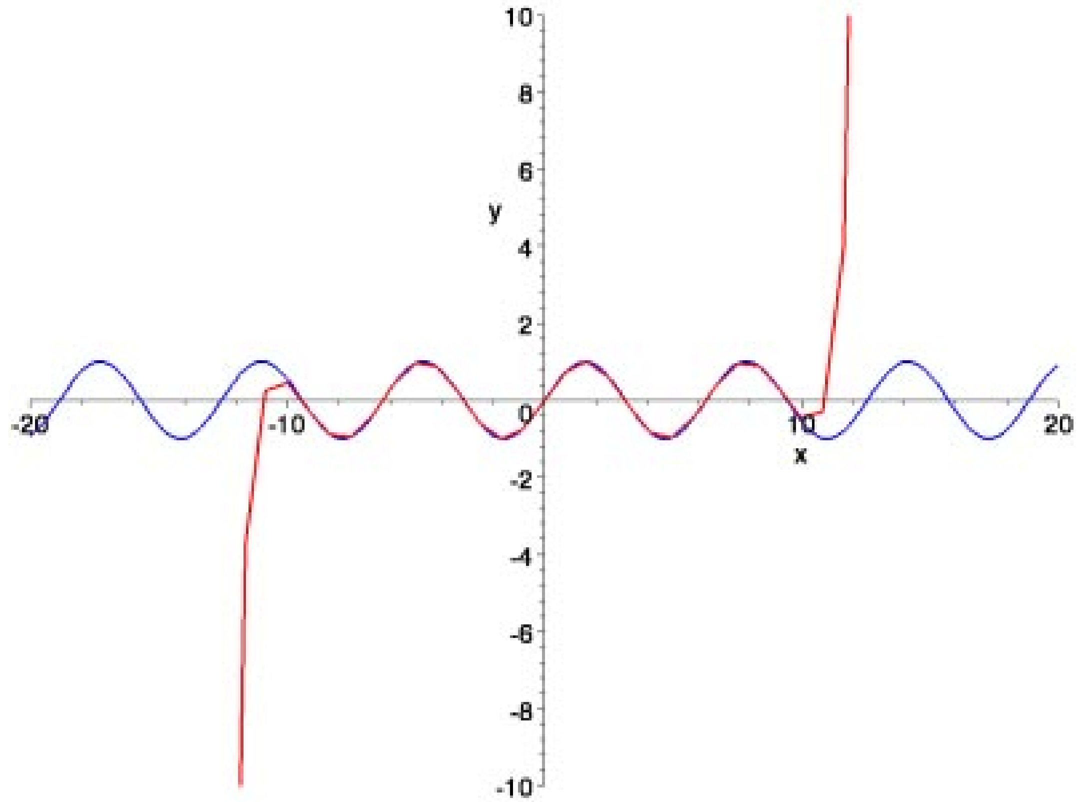
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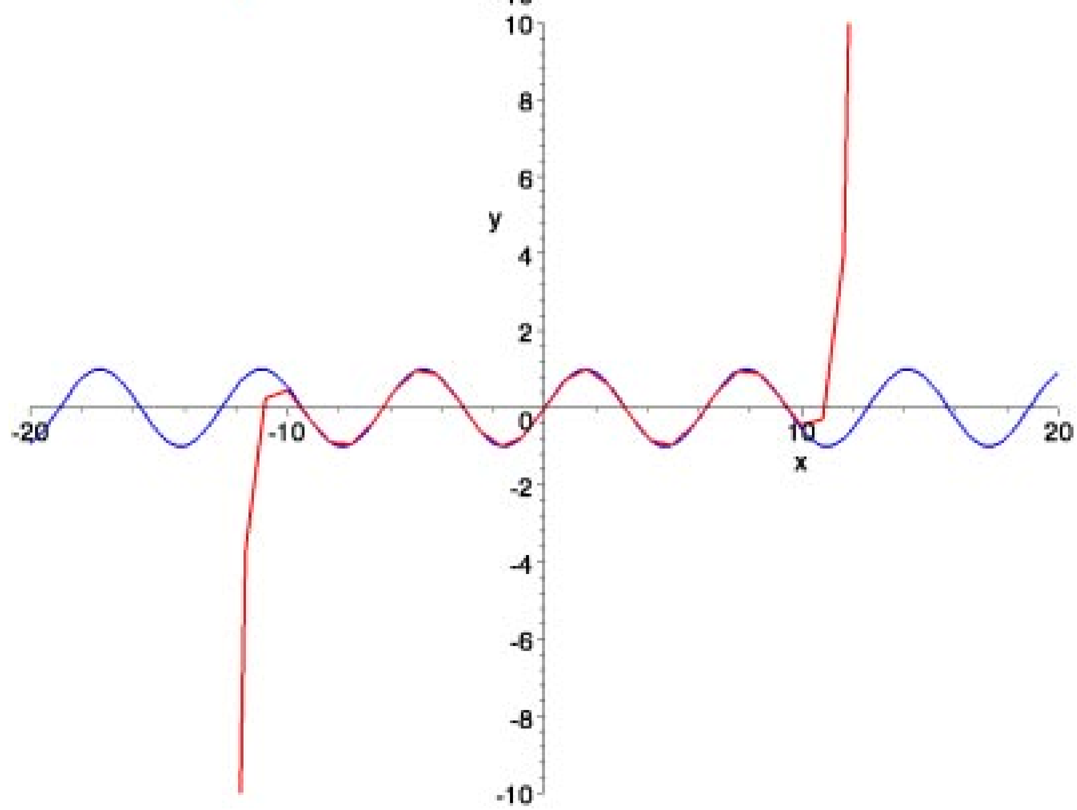
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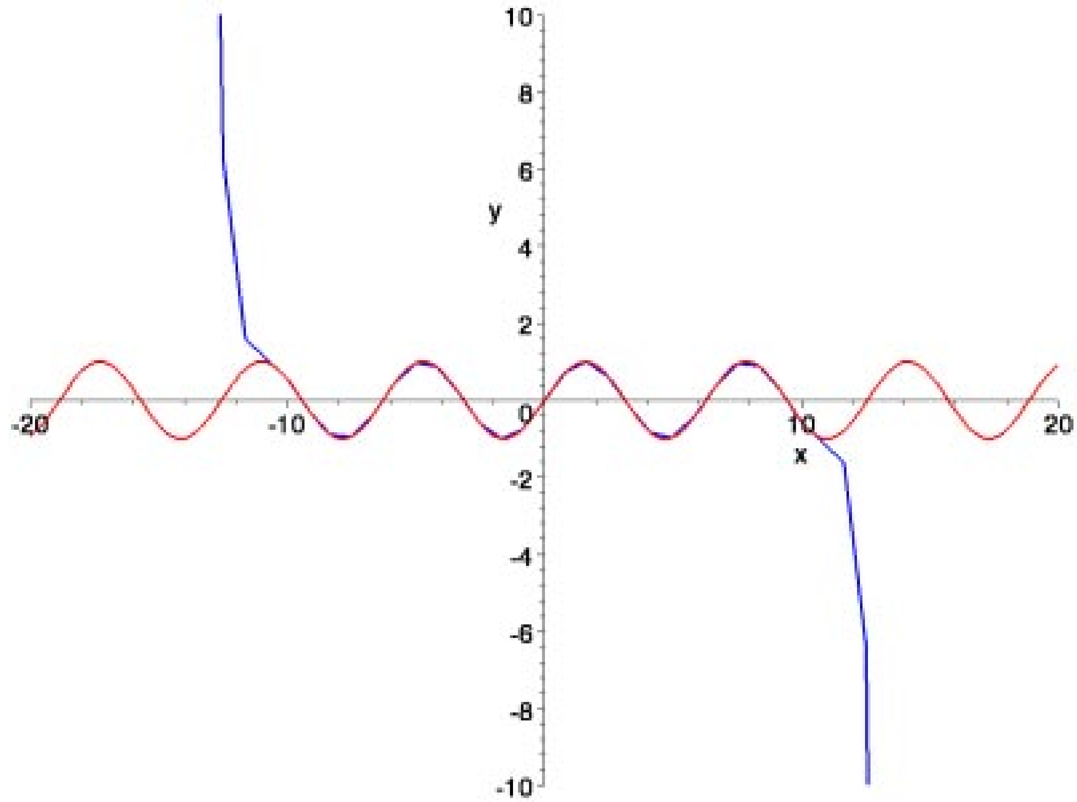
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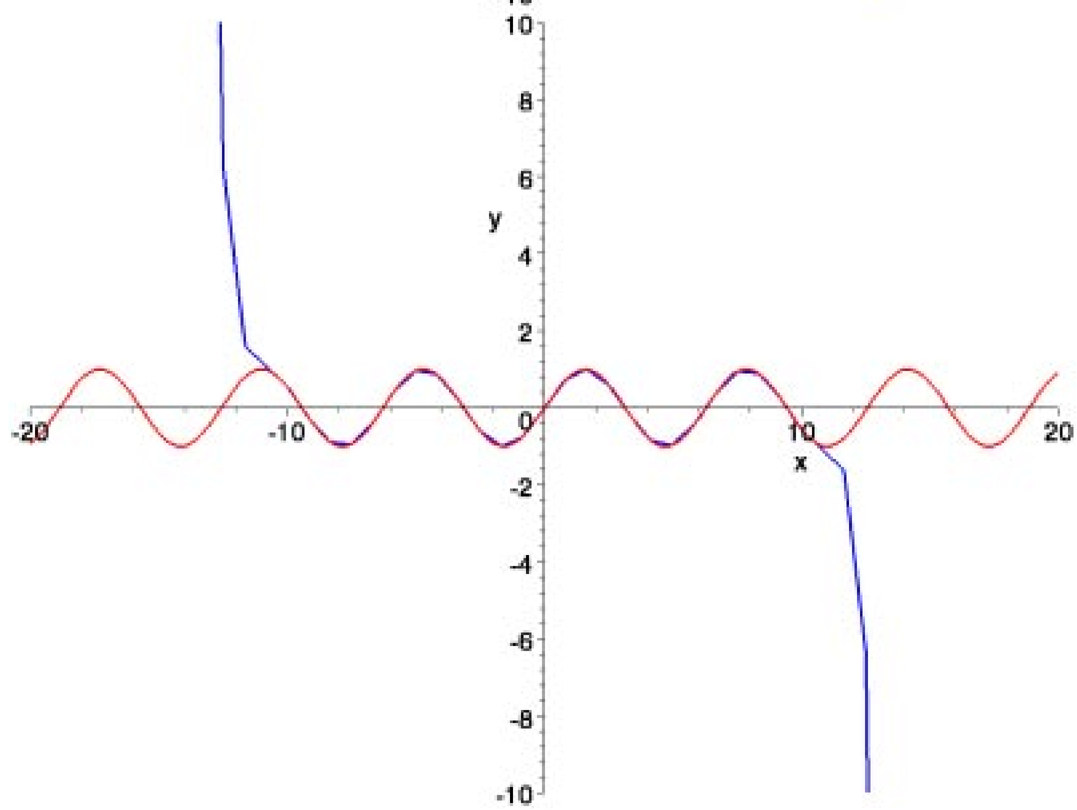
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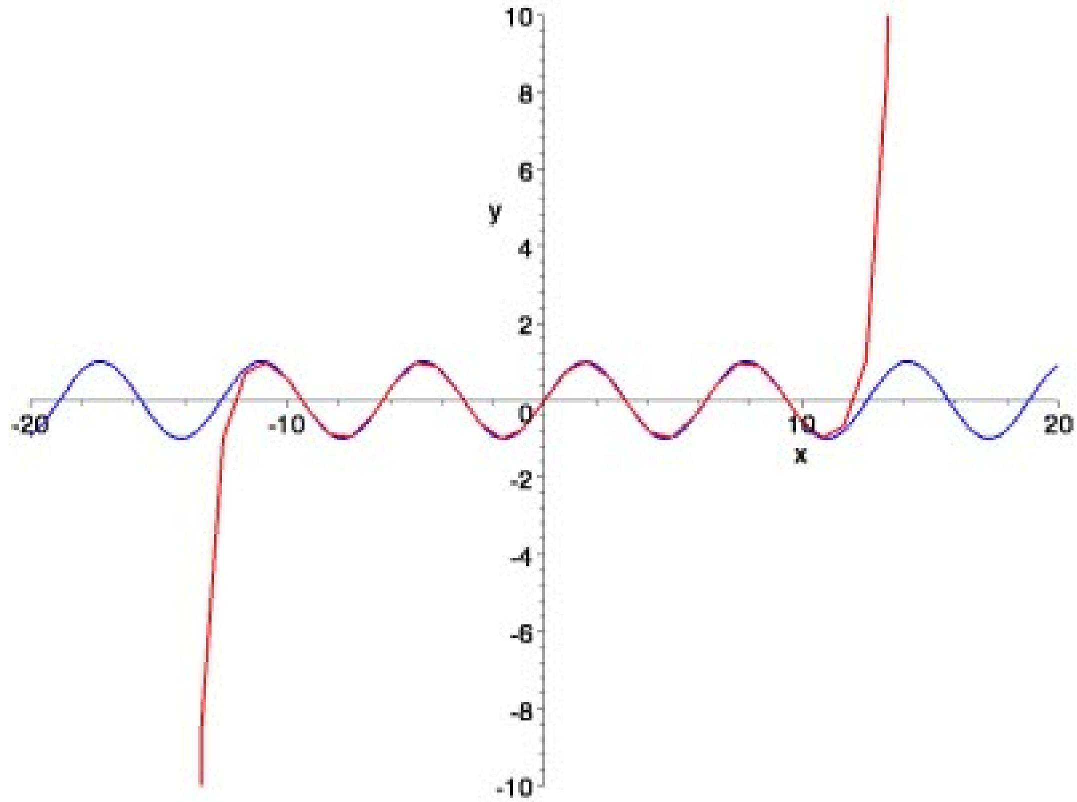
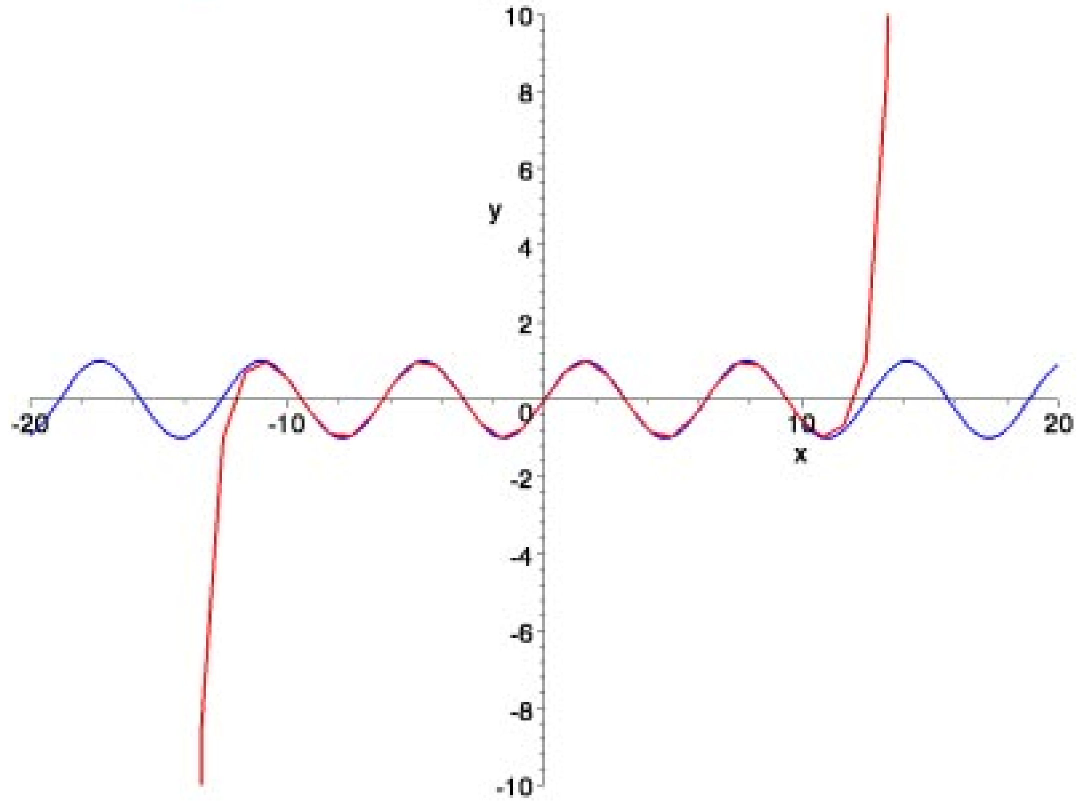
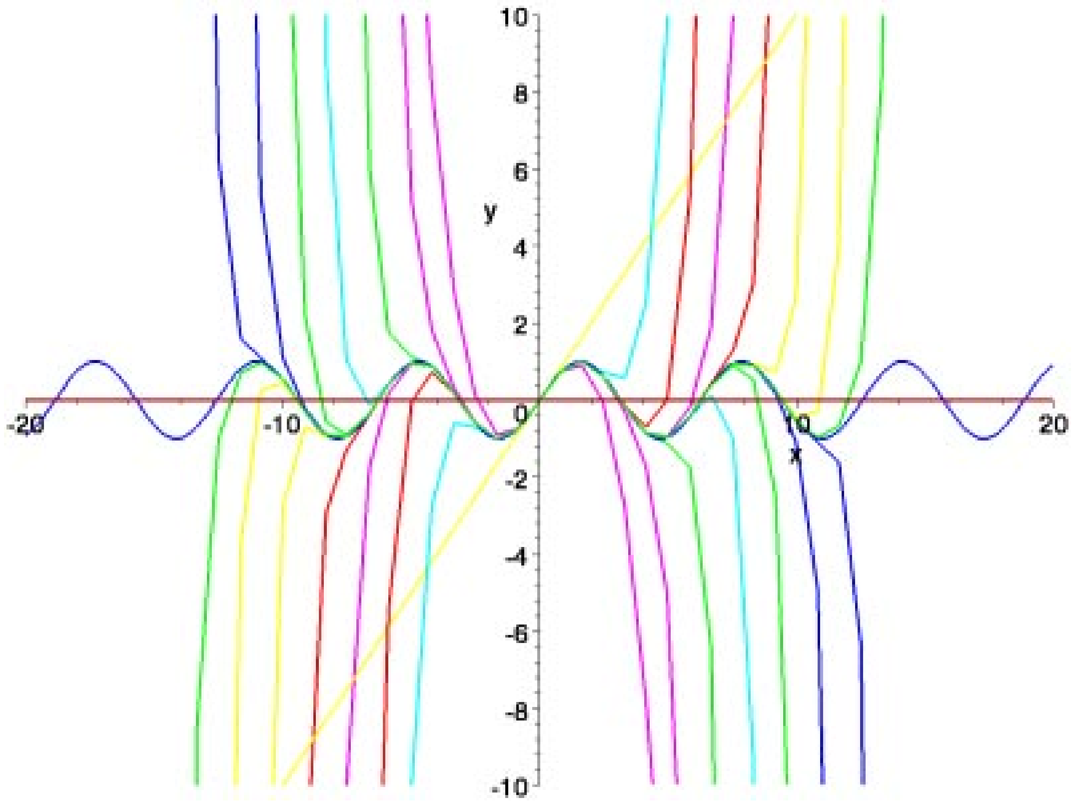




We notice that the higher the degree of the polynomial, the farther we can move away from 0 and still have a good approximation. Next we look at all the graphs together.

```
> plot({f,seq(P[n],n=0..30)},x=-20..20,y=-10..10,thickness=3);
```



What does it mean for a power series to converge for a given x? For $x = 5$, we look at the sequence $P_n(5)$ for n from 0 to 40.

```
> P:='P':
  for n from 0 to 40 do
  P[n](5)=evalf(subs(x=5,convert(taylor(f,x=0,n+1),polynom)));
  od;
```

$$P_0(5) = 0.$$

$$P_1(5) = 5.$$

$$P_2(5) = 5.$$

$$P_3(5) = -15.83333333$$

$$P_4(5) = -15.83333333$$

$$P_5(5) = 10.20833333$$

$$P_6(5) = 10.20833333$$

$$P_7(5) = -5.292658730$$

$$P_8(5) = -5.292658730$$

$$P_9(5) = .08963018078$$

$$P_{10}(5) = .08963018078$$

$$P_{11}(5) = -1.133617299$$

$$P_{12}(5) = -1.133617299$$

$$P_{13}(5) = -.9375840490$$

$$P_{14}(5) = -.9375840490$$

$$P_{15}(5) = -.9609213407$$

$$P_{16}(5) = -.9609213407$$
$$P_{17}(5) = -.9587763690$$
$$P_{18}(5) = -.9587763690$$
$$P_{19}(5) = -.9589331652$$
$$P_{20}(5) = -.9589331652$$
$$P_{21}(5) = -.9589238321$$
$$P_{22}(5) = -.9589238321$$
$$P_{23}(5) = -.9589242932$$
$$P_{24}(5) = -.9589242932$$
$$P_{25}(5) = -.9589242740$$
$$P_{26}(5) = -.9589242740$$
$$P_{27}(5) = -.9589242747$$
$$P_{28}(5) = -.9589242747$$
$$P_{29}(5) = -.9589242747$$
$$P_{30}(5) = -.9589242747$$
$$P_{31}(5) = -.9589242747$$
$$P_{32}(5) = -.9589242747$$
$$P_{33}(5) = -.9589242747$$
$$P_{34}(5) = -.9589242747$$
$$P_{35}(5) = -.9589242747$$
$$P_{36}(5) = -.9589242747$$
$$P_{37}(5) = -.9589242747$$
$$P_{38}(5) = -.9589242747$$
$$P_{39}(5) = -.9589242747$$
$$P_{40}(5) = -.9589242747$$

We see that this sequence settles in on one particular number as n gets "large." Just to check if it stays at that number for even larger n, let's find $P_{200}(5)$.

```
> evalf(subs(x=5,convert(taylor(f,x=0,201),polynom)));
```

$$-.9589242747$$

We take it that the **sequence of partial sums** converges to -.9589242747 to 10 decimal places, and take this limit to be $\mathrm{T}(5)$. but is this the same as $\sin(5)$?

```
> evalf(sin(5));
```

$$-.9589242747$$

It seems so. In general, for any power series $\mathrm{P}(x)$ and any given $x = a$, we say that $\mathrm{P}(x)$ converges to L for $x = a$ if $\lim_{n \to \infty} P_n(a) = L$. Now, if $\mathrm{P}(x)$ is a Taylor series for the functions $\mathrm{f}(x)$ we will be considering in this course (but not necessarily other functions), $\mathrm{f}(a) = L$ also, i.e., the Taylor series converges to the function value at the given point a. We note here that the Taylor series for $\mathbf{e}^x$, $\sin(x)$, and $\cos(x)$ converge for all x. Next let's consider the function $\mathrm{f}(x) = \frac{1}{1-x}$.

```
> restart:
```

```
> f:=1/(1-x);
```

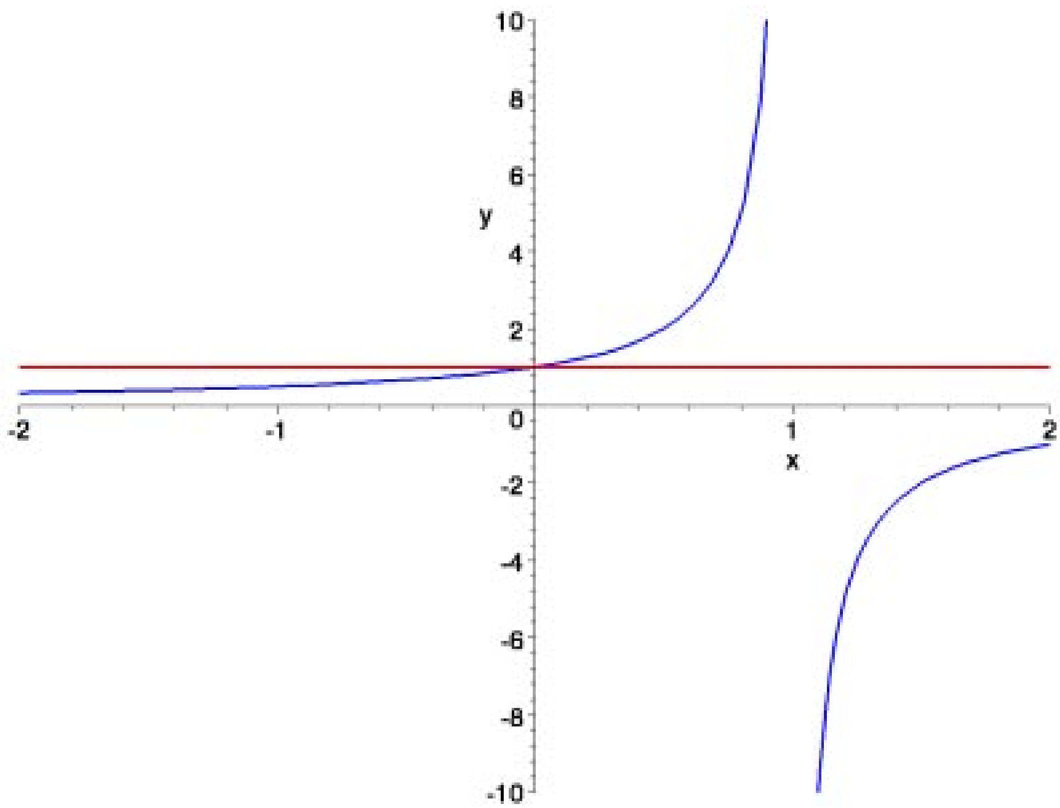
$$f := \frac{1}{1-x}$$

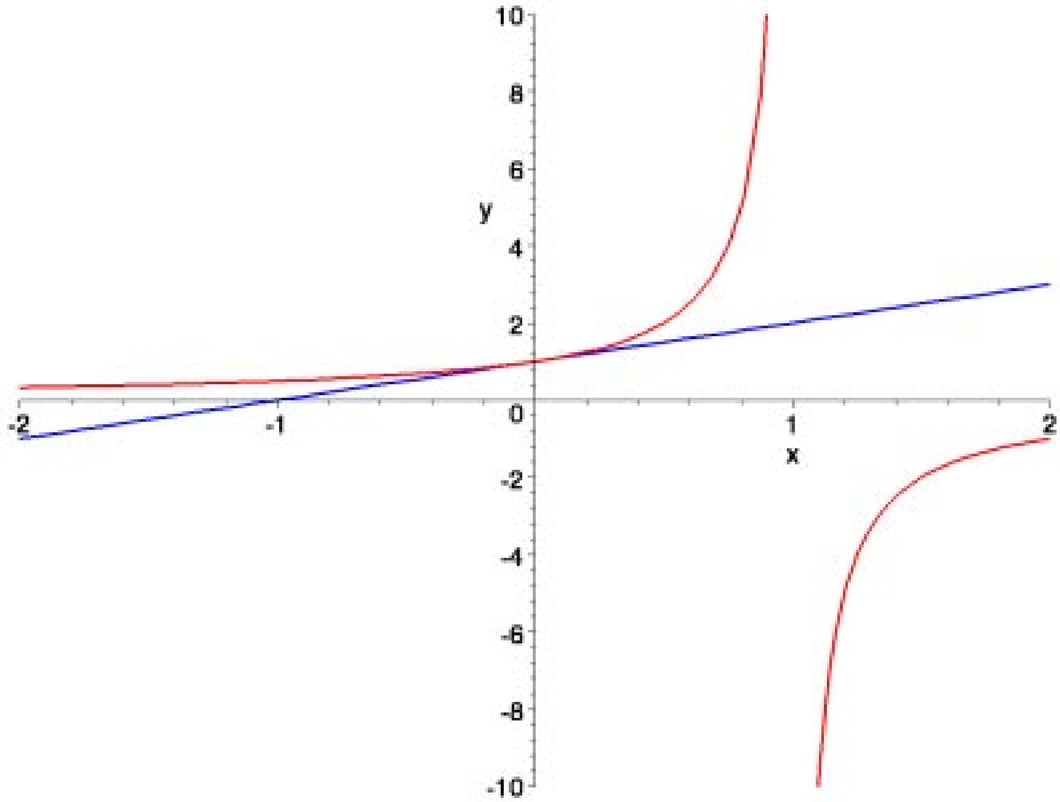
This function has as its Taylor series

$$\mathrm{T}(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots$$

We calculate and plot in succession the first 11 Taylor polynomials about $x = 0$.

```
> for n from 0 to 10 do
  P[n]:=convert(taylor(f,x=0,n+1),polynom):
  od:
> for n from 0 to 10 do
  plot({f,P[n]},x=-2..2,y=-10..10,color=[red,blue],thickness=3,discont=
  true):
  od;
```

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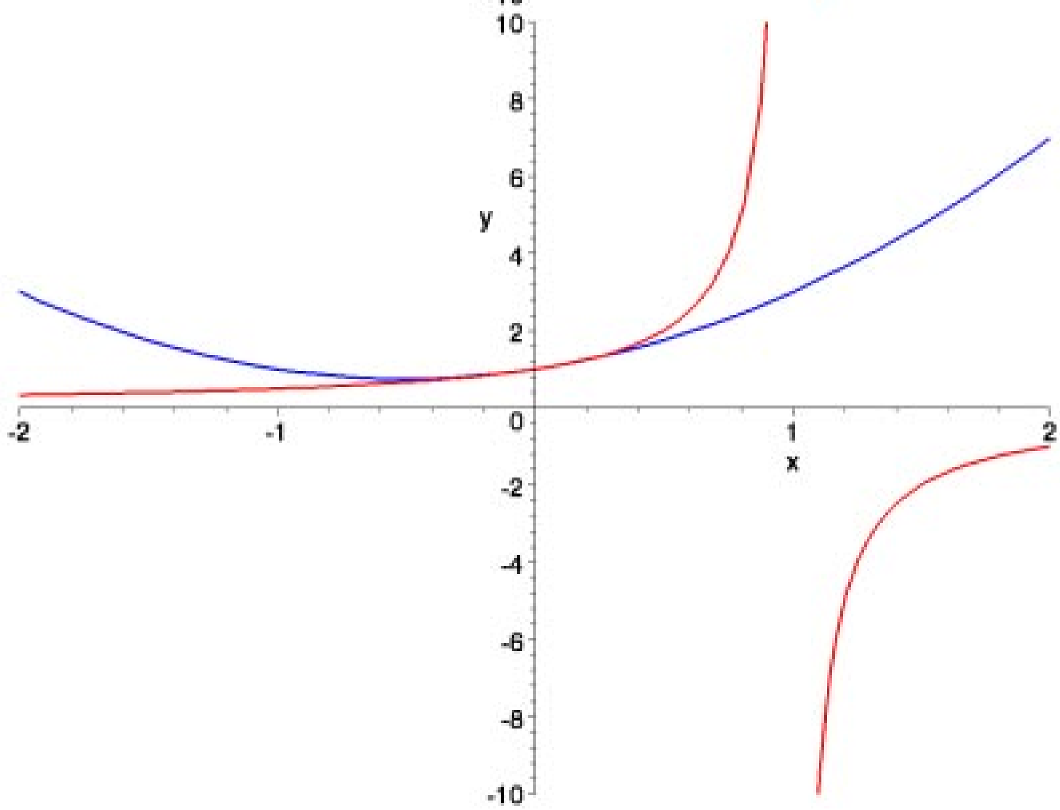
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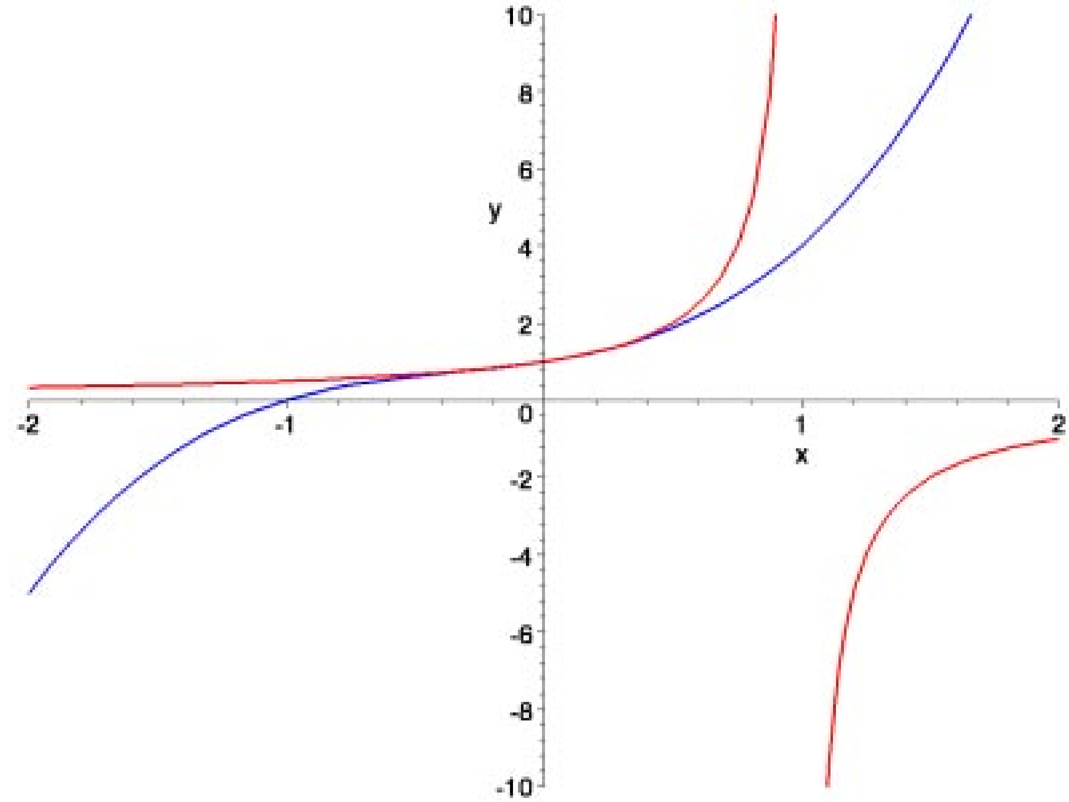

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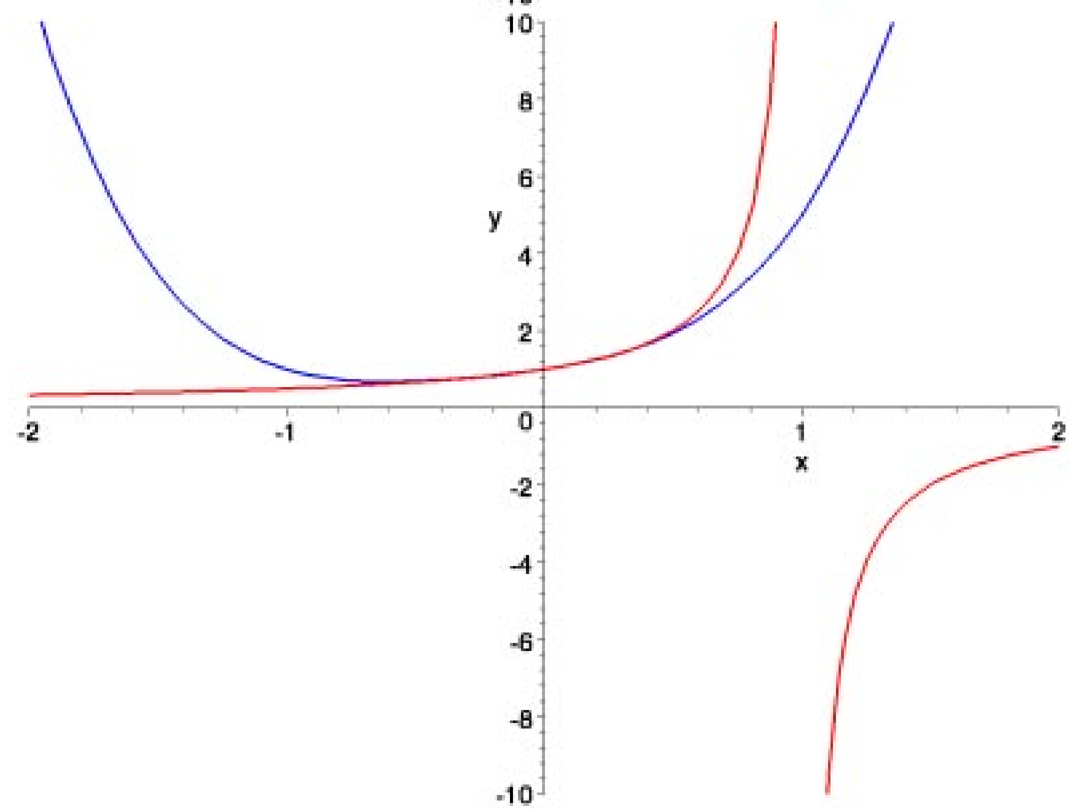

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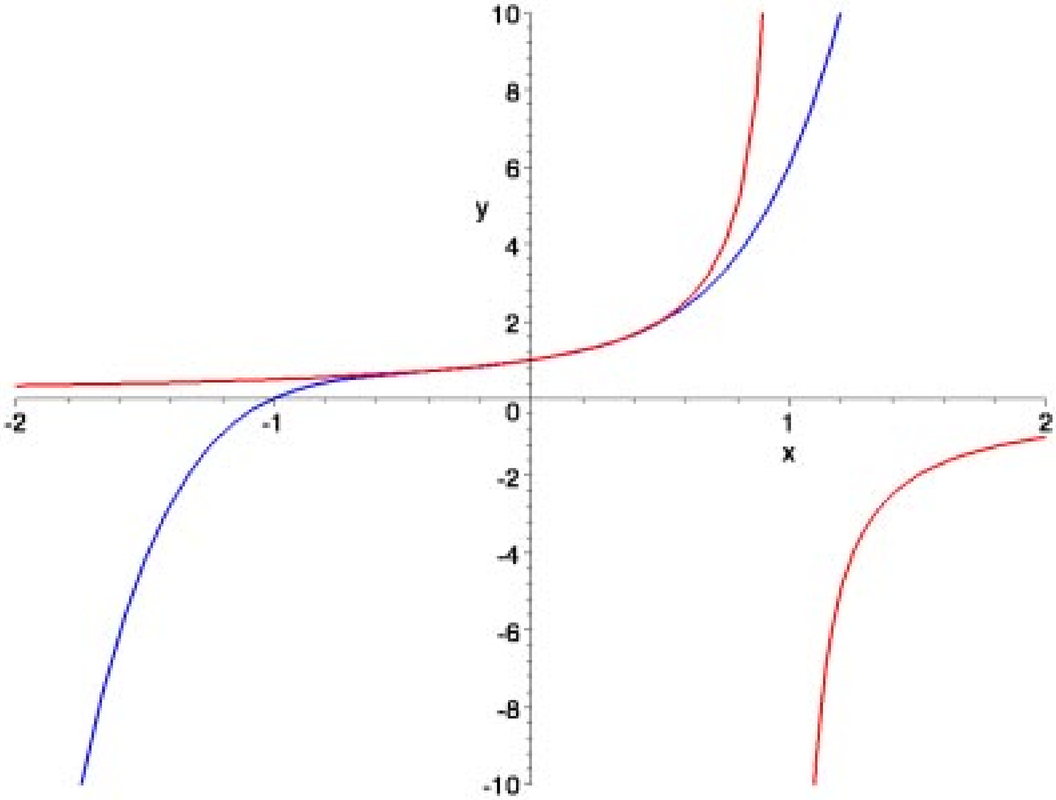

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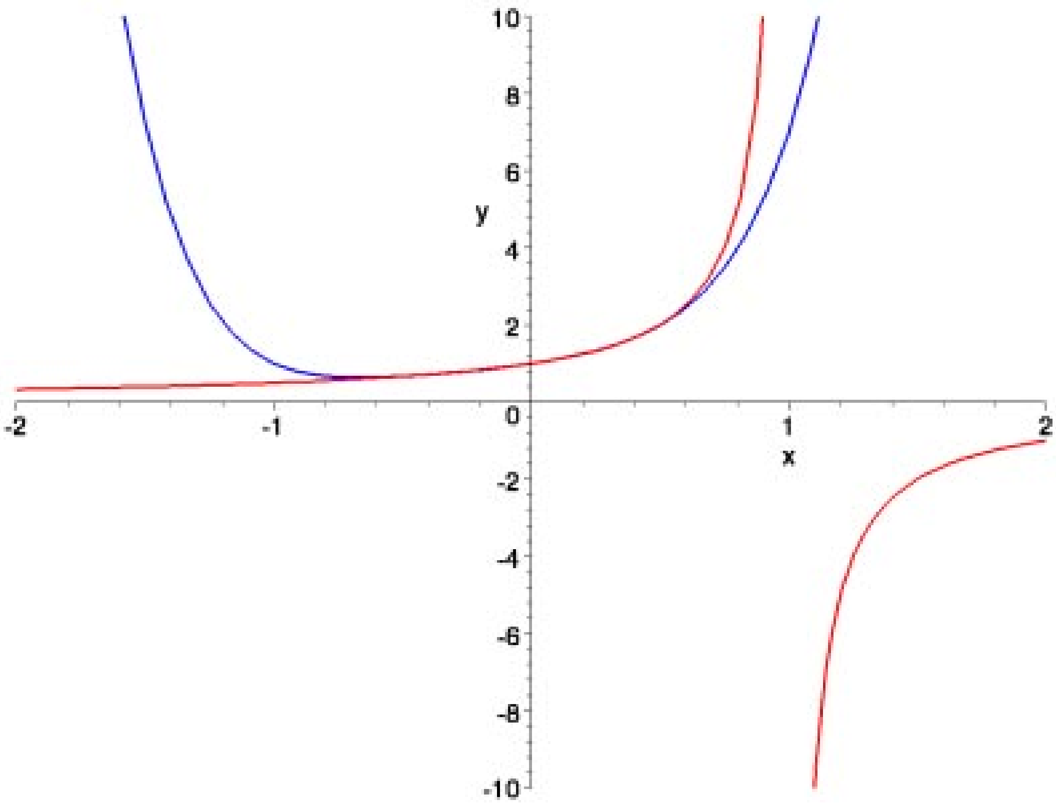

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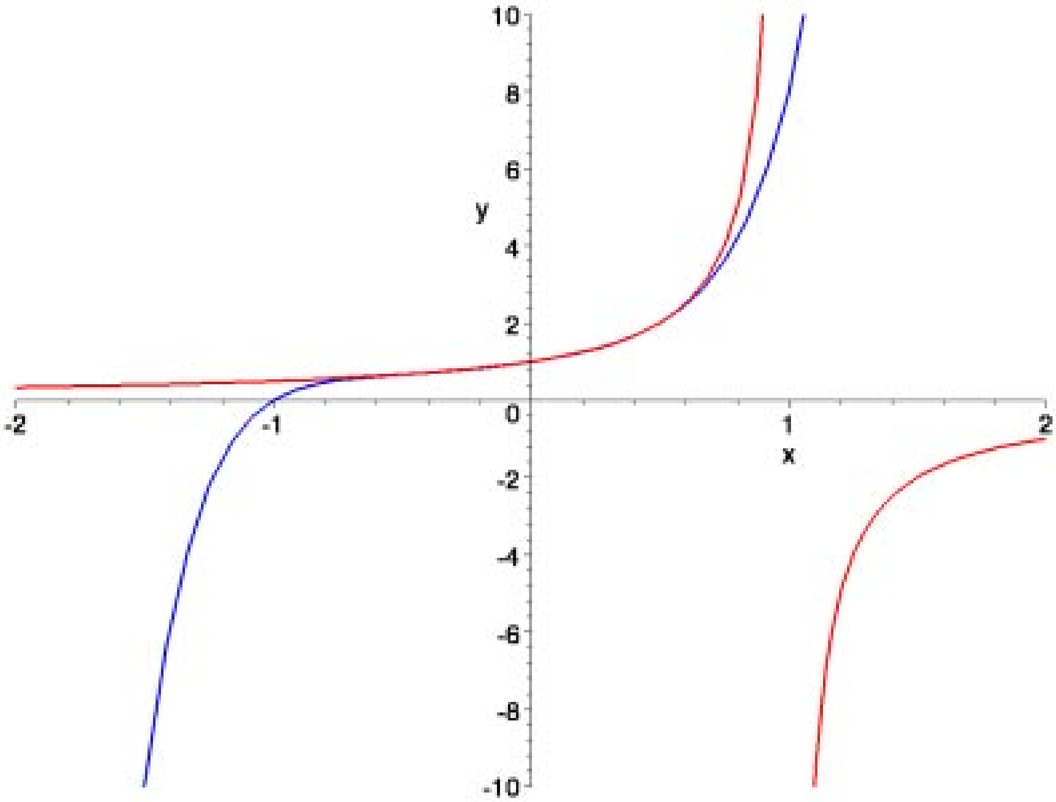
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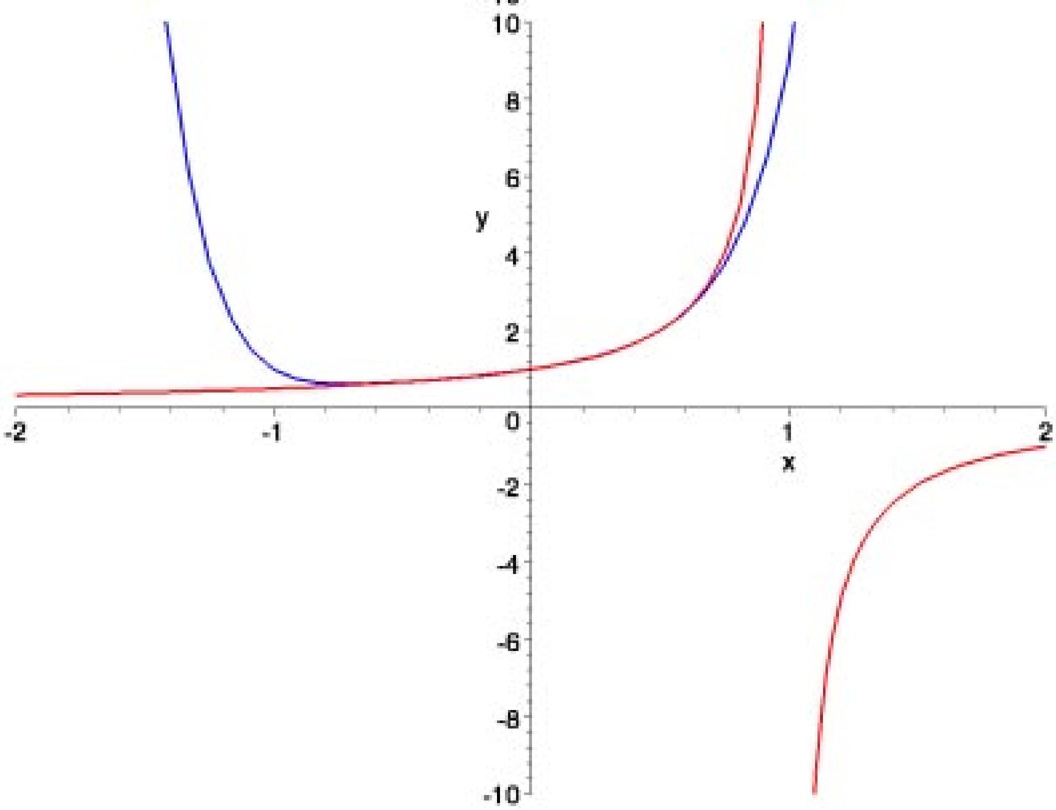
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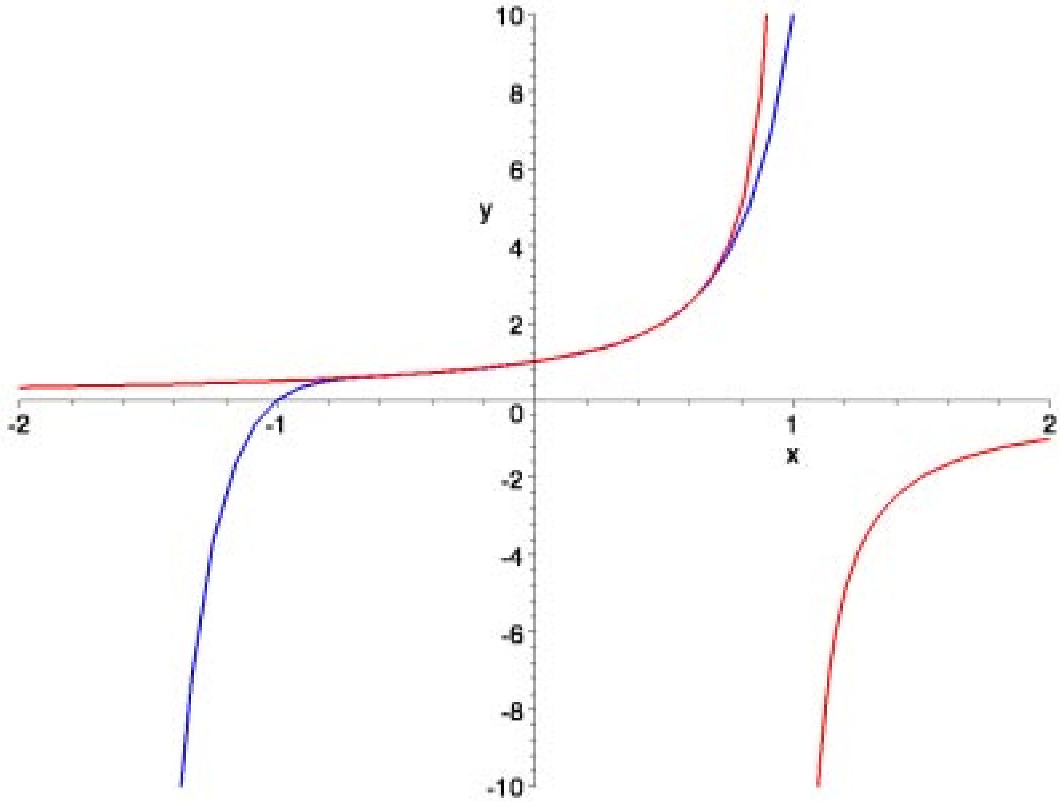
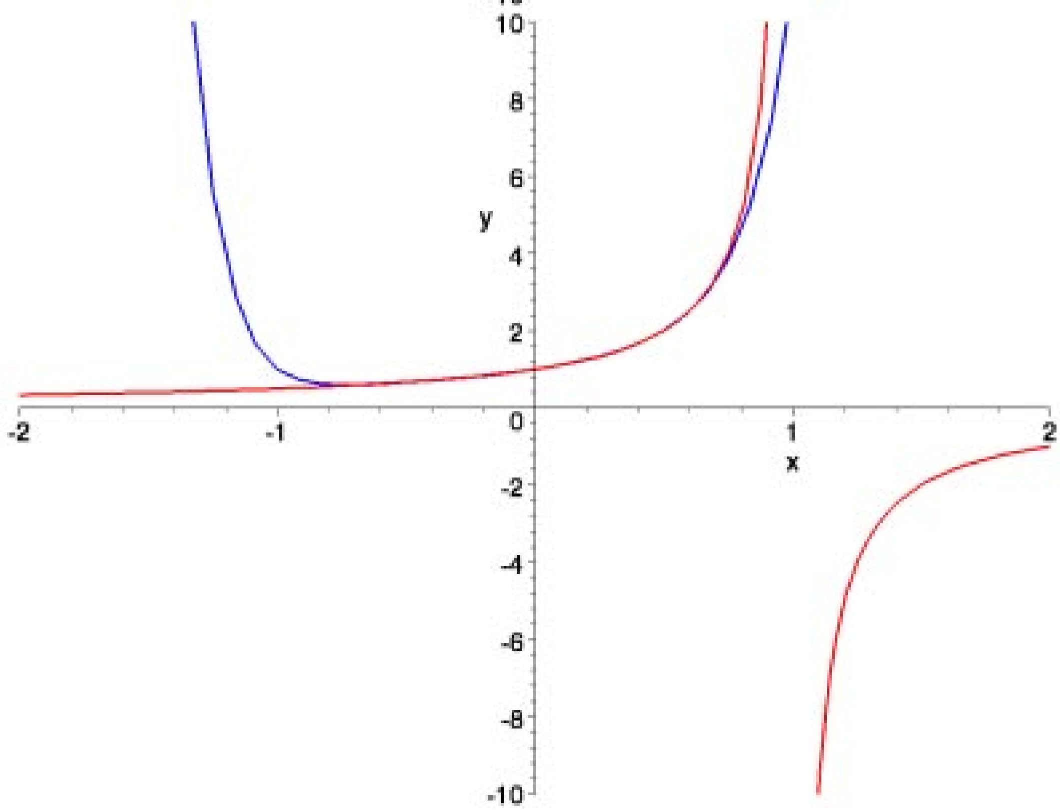
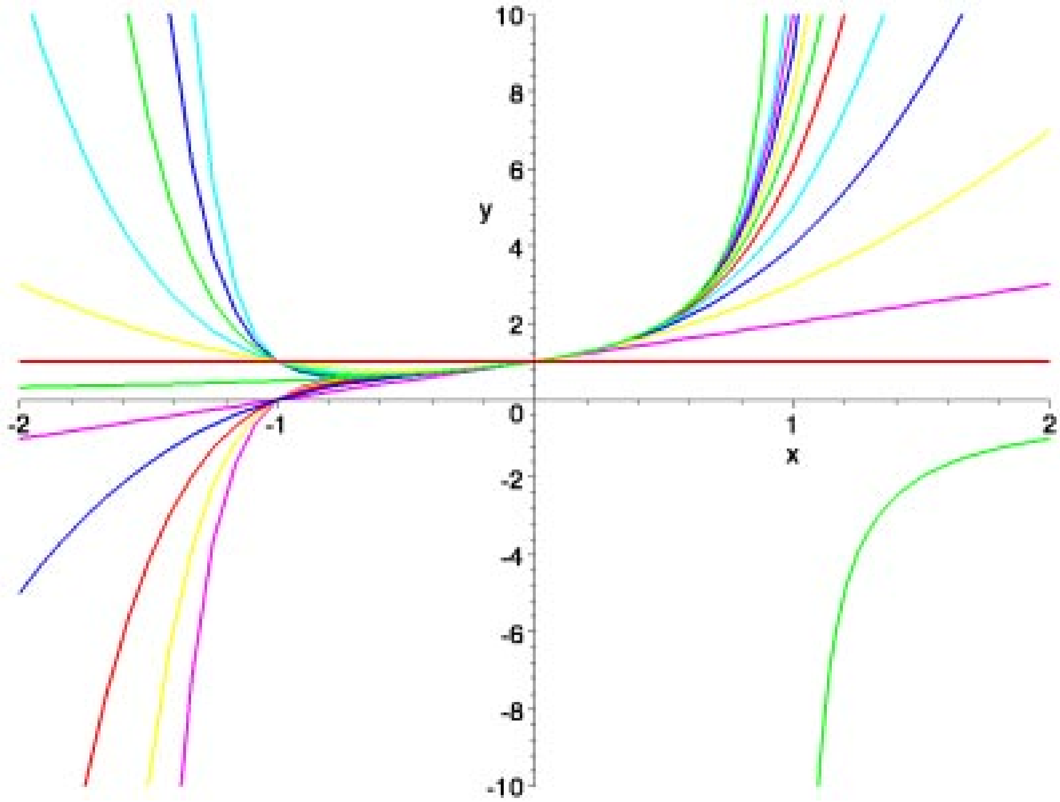
x





We see that our approximations are good between -1 and 1, but nowhere else. In fact, all ot the Taylor polynomials are positive to the right of 1, but f(x) is negative there. Let's look at all of the graphs together.

```
> plot({f,seq(P[n],n=0..10)},x=-2..2,y=-10..10,thickness=3,discont=true
  );
```



We have seen that some power series converge for all *x*. Others, such as this one, converge exactly for all *x* within some radius *R* of the center of the series, called the **radius of convergence** of the series. A final type of series converges only at a single point, its center. For our current series, let's illustrate its convergence at $x = .9$. We can easily calculate that f(.9) = 10.

```
> P:='P':
  for n from 0 to 220 do
  P[n](.9)=evalf(subs(x=.9,convert(taylor(f,x=0,n+1),polynom)));
  od;
```

$$P_0(.9) = 1.$$

$$P_1(.9) = 1.9$$

$$P_2(.9) = 2.71$$

$$P_3(.9) = 3.439$$

$$P_4(.9) = 4.0951$$

$$P_5(.9) = 4.68559$$

$$P_6(.9) = 5.217031$$

$$P_7(.9) = 5.6953279$$

$$P_8(.9) = 6.12579511$$

$$P_9(.9) = 6.513215599$$

$$P_{10}(.9) = 6.861894039$$

$$P_{11}(.9) = 7.175704635$$

$$P_{12}(.9) = 7.458134172$$

$$P_{13}(.9) = 7.712320755$$

$P_{14}(.9) = 7.941088680$

$P_{15}(.9) = 8.146979812$

$P_{16}(.9) = 8.332281831$

$P_{17}(.9) = 8.499053648$

$P_{18}(.9) = 8.649148283$

$P_{19}(.9) = 8.784233455$

$P_{20}(.9) = 8.905810110$

$P_{21}(.9) = 9.015229099$

$P_{22}(.9) = 9.113706189$

$P_{23}(.9) = 9.202335570$

$P_{24}(.9) = 9.282102013$

$P_{25}(.9) = 9.353891812$

$P_{26}(.9) = 9.418502630$

$P_{27}(.9) = 9.476652368$

$P_{28}(.9) = 9.528987130$

$P_{29}(.9) = 9.576088418$

$P_{30}(.9) = 9.618479576$

$P_{31}(.9) = 9.656631618$

$P_{32}(.9) = 9.690968456$

$P_{33}(.9) = 9.721871610$

$P_{34}(.9) = 9.749684449$

$P_{35}(.9) = 9.774716004$

$P_{36}(.9) = 9.797244404$

$P_{37}(.9) = 9.817519964$

$P_{38}(.9) = 9.835767968$

$P_{39}(.9) = 9.852191171$

$P_{40}(.9) = 9.866972054$

$P_{41}(.9) = 9.880274849$

$P_{42}(.9) = 9.892247364$

$P_{43}(.9) = 9.903022628$

$P_{44}(.9) = 9.912720365$

$P_{45}(.9) = 9.921448329$

$P_{46}(.9) = 9.929303496$

$P_{47}(.9) = 9.936373146$

$P_{48}(.9) = 9.942735831$

$P_{49}(.9) = 9.948462248$

$P_{50}(.9) = 9.953616023$

$P_{51}(.9) = 9.958254421$
$P_{52}(.9) = 9.962428979$
$P_{53}(.9) = 9.966186081$
$P_{54}(.9) = 9.969567473$
$P_{55}(.9) = 9.972610726$
$P_{56}(.9) = 9.975349653$
$P_{57}(.9) = 9.977814688$
$P_{58}(.9) = 9.980033218$
$P_{59}(.9) = 9.982029896$
$P_{60}(.9) = 9.983826906$
$P_{61}(.9) = 9.985444216$
$P_{62}(.9) = 9.986899794$
$P_{63}(.9) = 9.988209815$
$P_{64}(.9) = 9.989388833$
$P_{65}(.9) = 9.990449949$
$P_{66}(.9) = 9.991404954$
$P_{67}(.9) = 9.992264458$
$P_{68}(.9) = 9.993038012$
$P_{69}(.9) = 9.993734212$
$P_{70}(.9) = 9.994360791$
$P_{71}(.9) = 9.994924712$
$P_{72}(.9) = 9.995432241$
$P_{73}(.9) = 9.995889017$
$P_{74}(.9) = 9.996300115$
$P_{75}(.9) = 9.996670103$
$P_{76}(.9) = 9.997003093$
$P_{77}(.9) = 9.997302784$
$P_{78}(.9) = 9.997572506$
$P_{79}(.9) = 9.997815255$
$P_{80}(.9) = 9.998033730$
$P_{81}(.9) = 9.998230357$
$P_{82}(.9) = 9.998407321$
$P_{83}(.9) = 9.998566589$
$P_{84}(.9) = 9.998709929$
$P_{85}(.9) = 9.998838936$
$P_{86}(.9) = 9.998955042$
$P_{87}(.9) = 9.999059538$

$$P_{88}(.9) = 9.999153584$$
$$P_{89}(.9) = 9.999238225$$
$$P_{90}(.9) = 9.999314402$$
$$P_{91}(.9) = 9.999382962$$
$$P_{92}(.9) = 9.999444666$$
$$P_{93}(.9) = 9.999500199$$
$$P_{94}(.9) = 9.999550179$$
$$P_{95}(.9) = 9.999595161$$
$$P_{96}(.9) = 9.999635645$$
$$P_{97}(.9) = 9.999672080$$
$$P_{98}(.9) = 9.999704872$$
$$P_{99}(.9) = 9.999734385$$
$$P_{100}(.9) = 9.999760946$$
$$P_{101}(.9) = 9.999784851$$
$$P_{102}(.9) = 9.999806367$$
$$P_{103}(.9) = 9.999825730$$
$$P_{104}(.9) = 9.999843157$$
$$P_{105}(.9) = 9.999858841$$
$$P_{106}(.9) = 9.999872957$$
$$P_{107}(.9) = 9.999885661$$
$$P_{108}(.9) = 9.999897095$$
$$P_{109}(.9) = 9.999907385$$
$$P_{110}(.9) = 9.999916646$$
$$P_{111}(.9) = 9.999924981$$
$$P_{112}(.9) = 9.999932483$$
$$P_{113}(.9) = 9.999939235$$
$$P_{114}(.9) = 9.999945311$$
$$P_{115}(.9) = 9.999950780$$
$$P_{116}(.9) = 9.999955702$$
$$P_{117}(.9) = 9.999960132$$
$$P_{118}(.9) = 9.999964119$$
$$P_{119}(.9) = 9.999967707$$
$$P_{120}(.9) = 9.999970935$$
$$P_{121}(.9) = 9.999973841$$
$$P_{122}(.9) = 9.999976457$$
$$P_{123}(.9) = 9.999978811$$
$$P_{124}(.9) = 9.999980930$$

$P_{125}(.9) = 9.999982837$
$P_{126}(.9) = 9.999984553$
$P_{127}(.9) = 9.999986098$
$P_{128}(.9) = 9.999987488$
$P_{129}(.9) = 9.999988739$
$P_{130}(.9) = 9.999989865$
$P_{131}(.9) = 9.999990879$
$P_{132}(.9) = 9.999991791$
$P_{133}(.9) = 9.999992612$
$P_{134}(.9) = 9.999993351$
$P_{135}(.9) = 9.999994016$
$P_{136}(.9) = 9.999994614$
$P_{137}(.9) = 9.999995153$
$P_{138}(.9) = 9.999995638$
$P_{139}(.9) = 9.999996074$
$P_{140}(.9) = 9.999996467$
$P_{141}(.9) = 9.999996820$
$P_{142}(.9) = 9.999997138$
$P_{143}(.9) = 9.999997424$
$P_{144}(.9) = 9.999997682$
$P_{145}(.9) = 9.999997914$
$P_{146}(.9) = 9.999998123$
$P_{147}(.9) = 9.999998311$
$P_{148}(.9) = 9.999998480$
$P_{149}(.9) = 9.999998632$
$P_{150}(.9) = 9.999998769$
$P_{151}(.9) = 9.999998892$
$P_{152}(.9) = 9.999999002$
$P_{153}(.9) = 9.999999102$
$P_{154}(.9) = 9.999999192$
$P_{155}(.9) = 9.999999274$
$P_{156}(.9) = 9.999999347$
$P_{157}(.9) = 9.999999412$
$P_{158}(.9) = 9.999999471$
$P_{159}(.9) = 9.999999524$
$P_{160}(.9) = 9.999999572$
$P_{161}(.9) = 9.999999615$

$$P_{162}(.9) = 9.999999653$$
$$P_{163}(.9) = 9.999999689$$
$$P_{164}(.9) = 9.999999720$$
$$P_{165}(.9) = 9.999999748$$
$$P_{166}(.9) = 9.999999773$$
$$P_{167}(.9) = 9.999999796$$
$$P_{168}(.9) = 9.999999817$$
$$P_{169}(.9) = 9.999999835$$
$$P_{170}(.9) = 9.999999852$$
$$P_{171}(.9) = 9.999999867$$
$$P_{172}(.9) = 9.999999879$$
$$P_{173}(.9) = 9.999999891$$
$$P_{174}(.9) = 9.999999902$$
$$P_{175}(.9) = 9.999999912$$
$$P_{176}(.9) = 9.999999921$$
$$P_{177}(.9) = 9.999999929$$
$$P_{178}(.9) = 9.999999936$$
$$P_{179}(.9) = 9.999999942$$
$$P_{180}(.9) = 9.999999948$$
$$P_{181}(.9) = 9.999999953$$
$$P_{182}(.9) = 9.999999958$$
$$P_{183}(.9) = 9.999999962$$
$$P_{184}(.9) = 9.999999966$$
$$P_{185}(.9) = 9.999999969$$
$$P_{186}(.9) = 9.999999972$$
$$P_{187}(.9) = 9.999999975$$
$$P_{188}(.9) = 9.999999977$$
$$P_{189}(.9) = 9.999999979$$
$$P_{190}(.9) = 9.999999981$$
$$P_{191}(.9) = 9.999999983$$
$$P_{192}(.9) = 9.999999985$$
$$P_{193}(.9) = 9.999999986$$
$$P_{194}(.9) = 9.999999987$$
$$P_{195}(.9) = 9.999999988$$
$$P_{196}(.9) = 9.999999989$$
$$P_{197}(.9) = 9.999999991$$
$$P_{198}(.9) = 9.999999992$$

$$P_{199}(.9) = 9.999999993$$
$$P_{200}(.9) = 9.999999994$$
$$P_{201}(.9) = 9.999999995$$
$$P_{202}(.9) = 9.999999996$$
$$P_{203}(.9) = 9.999999997$$
$$P_{204}(.9) = 9.999999997$$
$$P_{205}(.9) = 9.999999997$$
$$P_{206}(.9) = 9.999999997$$
$$P_{207}(.9) = 9.999999997$$
$$P_{208}(.9) = 9.999999997$$
$$P_{209}(.9) = 9.999999997$$
$$P_{210}(.9) = 9.999999997$$
$$P_{211}(.9) = 9.999999997$$
$$P_{212}(.9) = 9.999999997$$
$$P_{213}(.9) = 9.999999997$$
$$P_{214}(.9) = 9.999999997$$
$$P_{215}(.9) = 9.999999997$$
$$P_{216}(.9) = 9.999999997$$
$$P_{217}(.9) = 9.999999997$$
$$P_{218}(.9) = 9.999999997$$
$$P_{219}(.9) = 9.999999997$$
$$P_{220}(.9) = 9.999999997$$

This certainly appears to be a convergent sequence, so we conclude the series converges for $x = .9$.
Now let's try the same for $x = -1$ and $x = 1$.

```
> for n from 0 to 50 do
  P[n](-1)=evalf(subs(x=-1,convert(taylor(f,x=0,n+1),polynom)));
  od;
```

$$P_{0}(-1) = 1.$$
$$P_{1}(-1) = 0.$$
$$P_{2}(-1) = 1.$$
$$P_{3}(-1) = 0.$$
$$P_{4}(-1) = 1.$$
$$P_{5}(-1) = 0.$$
$$P_{6}(-1) = 1.$$
$$P_{7}(-1) = 0.$$
$$P_{8}(-1) = 1.$$
$$P_{9}(-1) = 0.$$
$$P_{10}(-1) = 1.$$
$$P_{11}(-1) = 0.$$

$$P_{12}(-1) = 1.$$
$$P_{13}(-1) = 0.$$
$$P_{14}(-1) = 1.$$
$$P_{15}(-1) = 0.$$
$$P_{16}(-1) = 1.$$
$$P_{17}(-1) = 0.$$
$$P_{18}(-1) = 1.$$
$$P_{19}(-1) = 0.$$
$$P_{20}(-1) = 1.$$
$$P_{21}(-1) = 0.$$
$$P_{22}(-1) = 1.$$
$$P_{23}(-1) = 0.$$
$$P_{24}(-1) = 1.$$
$$P_{25}(-1) = 0.$$
$$P_{26}(-1) = 1.$$
$$P_{27}(-1) = 0.$$
$$P_{28}(-1) = 1.$$
$$P_{29}(-1) = 0.$$
$$P_{30}(-1) = 1.$$
$$P_{31}(-1) = 0.$$
$$P_{32}(-1) = 1.$$
$$P_{33}(-1) = 0.$$
$$P_{34}(-1) = 1.$$
$$P_{35}(-1) = 0.$$
$$P_{36}(-1) = 1.$$
$$P_{37}(-1) = 0.$$
$$P_{38}(-1) = 1.$$
$$P_{39}(-1) = 0.$$
$$P_{40}(-1) = 1.$$
$$P_{41}(-1) = 0.$$
$$P_{42}(-1) = 1.$$
$$P_{43}(-1) = 0.$$
$$P_{44}(-1) = 1.$$
$$P_{45}(-1) = 0.$$
$$P_{46}(-1) = 1.$$
$$P_{47}(-1) = 0.$$
$$P_{48}(-1) = 1.$$

$$P_{49}(-1) = 0.$$
$$P_{50}(-1) = 1.$$

```
> for n from 0 to 50 do
P[n](1)=evalf(subs(x=1,convert(taylor(f,x=0,n+1),polynom)));
od;
```

$$P_{0}(1) = 1.$$
$$P_{1}(1) = 2.$$
$$P_{2}(1) = 3.$$
$$P_{3}(1) = 4.$$
$$P_{4}(1) = 5.$$
$$P_{5}(1) = 6.$$
$$P_{6}(1) = 7.$$
$$P_{7}(1) = 8.$$
$$P_{8}(1) = 9.$$
$$P_{9}(1) = 10.$$
$$P_{10}(1) = 11.$$
$$P_{11}(1) = 12.$$
$$P_{12}(1) = 13.$$
$$P_{13}(1) = 14.$$
$$P_{14}(1) = 15.$$
$$P_{15}(1) = 16.$$
$$P_{16}(1) = 17.$$
$$P_{17}(1) = 18.$$
$$P_{18}(1) = 19.$$
$$P_{19}(1) = 20.$$
$$P_{20}(1) = 21.$$
$$P_{21}(1) = 22.$$
$$P_{22}(1) = 23.$$
$$P_{23}(1) = 24.$$
$$P_{24}(1) = 25.$$
$$P_{25}(1) = 26.$$
$$P_{26}(1) = 27.$$
$$P_{27}(1) = 28.$$
$$P_{28}(1) = 29.$$
$$P_{29}(1) = 30.$$
$$P_{30}(1) = 31.$$
$$P_{31}(1) = 32.$$
$$P_{32}(1) = 33.$$

$$P_{33}(1) = 34.$$
$$P_{34}(1) = 35.$$
$$P_{35}(1) = 36.$$
$$P_{36}(1) = 37.$$
$$P_{37}(1) = 38.$$
$$P_{38}(1) = 39.$$
$$P_{39}(1) = 40.$$
$$P_{40}(1) = 41.$$
$$P_{41}(1) = 42.$$
$$P_{42}(1) = 43.$$
$$P_{43}(1) = 44.$$
$$P_{44}(1) = 45.$$
$$P_{45}(1) = 46.$$
$$P_{46}(1) = 47.$$
$$P_{47}(1) = 48.$$
$$P_{48}(1) = 49.$$
$$P_{49}(1) = 50.$$
$$P_{50}(1) = 51.$$

Quite clearly, there is no convergence here. The radius of convergence for $f(x) = \frac{1}{1-x}$ is 1, which tells us that the series converges for x between -1 and 1. But one may or may not have convergence at the end points of the interval of convergence. Here we don't. Finally, let's look at -1.01, which is outside of the interval of convergence. We first find $f(-1.01)$.

```
> 1/(1-(-1.01));
```

$$.4975124378$$

```
> for n from 0 to 100 do
  P[n](-1.01)=evalf(subs(x=-1.01,convert(taylor(f,x=0,n+1),polynom)));
  od;
```

$$P_{0}(-1.01) = 1.$$
$$P_{1}(-1.01) = -.01$$
$$P_{2}(-1.01) = 1.0101$$
$$P_{3}(-1.01) = -.020201$$
$$P_{4}(-1.01) = 1.02040301$$
$$P_{5}(-1.01) = -.030607040$$
$$P_{6}(-1.01) = 1.030913111$$
$$P_{7}(-1.01) = -.041222241$$
$$P_{8}(-1.01) = 1.041634465$$
$$P_{9}(-1.01) = -.052050808$$
$$P_{10}(-1.01) = 1.052571317$$
$$P_{11}(-1.01) = -.063097030$$

$$P_{12}(-1.01) = 1.063728000$$
$$P_{13}(-1.01) = -.074365280$$
$$P_{14}(-1.01) = 1.075108933$$
$$P_{15}(-1.01) = -.085860022$$
$$P_{16}(-1.01) = 1.086718623$$
$$P_{17}(-1.01) = -.097585808$$
$$P_{18}(-1.01) = 1.098561668$$
$$P_{19}(-1.01) = -.109547282$$
$$P_{20}(-1.01) = 1.110642758$$
$$P_{21}(-1.01) = -.121749182$$
$$P_{22}(-1.01) = 1.122966678$$
$$P_{23}(-1.01) = -.134196340$$
$$P_{24}(-1.01) = 1.135538309$$
$$P_{25}(-1.01) = -.146893686$$
$$P_{26}(-1.01) = 1.148362629$$
$$P_{27}(-1.01) = -.159846249$$
$$P_{28}(-1.01) = 1.161444718$$
$$P_{29}(-1.01) = -.173059159$$
$$P_{30}(-1.01) = 1.174789756$$
$$P_{31}(-1.01) = -.186537648$$
$$P_{32}(-1.01) = 1.188403031$$
$$P_{33}(-1.01) = -.200287054$$
$$P_{34}(-1.01) = 1.202289932$$
$$P_{35}(-1.01) = -.214312824$$
$$P_{36}(-1.01) = 1.216455960$$
$$P_{37}(-1.01) = -.228620511$$
$$P_{38}(-1.01) = 1.230906725$$
$$P_{39}(-1.01) = -.243215784$$
$$P_{40}(-1.01) = 1.245647950$$
$$P_{41}(-1.01) = -.258104421$$
$$P_{42}(-1.01) = 1.260685474$$
$$P_{43}(-1.01) = -.273292320$$
$$P_{44}(-1.01) = 1.276025251$$
$$P_{45}(-1.01) = -.288785496$$
$$P_{46}(-1.01) = 1.291673359$$
$$P_{47}(-1.01) = -.304590084$$
$$P_{48}(-1.01) = 1.307635994$$

$$P_{49}(-1.01) = -.320712344$$
$$P_{50}(-1.01) = 1.323919478$$
$$P_{51}(-1.01) = -.337158662$$
$$P_{52}(-1.01) = 1.340530259$$
$$P_{53}(-1.01) = -.353935552$$
$$P_{54}(-1.01) = 1.357474917$$
$$P_{55}(-1.01) = -.371049656$$
$$P_{56}(-1.01) = 1.374760163$$
$$P_{57}(-1.01) = -.388507754$$
$$P_{58}(-1.01) = 1.392392843$$
$$P_{59}(-1.01) = -.406316760$$
$$P_{60}(-1.01) = 1.410379939$$
$$P_{61}(-1.01) = -.424483727$$
$$P_{62}(-1.01) = 1.428728575$$
$$P_{63}(-1.01) = -.443015850$$
$$P_{64}(-1.01) = 1.447446020$$
$$P_{65}(-1.01) = -.461920468$$
$$P_{66}(-1.01) = 1.466539685$$
$$P_{67}(-1.01) = -.481205070$$
$$P_{68}(-1.01) = 1.486017132$$
$$P_{69}(-1.01) = -.500877292$$
$$P_{70}(-1.01) = 1.505886076$$
$$P_{71}(-1.01) = -.520944926$$
$$P_{72}(-1.01) = 1.526154386$$
$$P_{73}(-1.01) = -.541415919$$
$$P_{74}(-1.01) = 1.546830089$$
$$P_{75}(-1.01) = -.562298379$$
$$P_{76}(-1.01) = 1.567921374$$
$$P_{77}(-1.01) = -.583600577$$
$$P_{78}(-1.01) = 1.589436593$$
$$P_{79}(-1.01) = -.605330949$$
$$P_{80}(-1.01) = 1.611384272$$
$$P_{81}(-1.01) = -.627498101$$
$$P_{82}(-1.01) = 1.633773092$$
$$P_{83}(-1.01) = -.650110813$$
$$P_{84}(-1.01) = 1.656611931$$
$$P_{85}(-1.01) = -.673178041$$

$$P_{86}(-1.01) = 1.679909830$$
$$P_{87}(-1.01) = -.696708920$$
$$P_{88}(-1.01) = 1.703676017$$
$$P_{89}(-1.01) = -.720712770$$
$$P_{90}(-1.01) = 1.727919905$$
$$P_{91}(-1.01) = -.745199096$$
$$P_{92}(-1.01) = 1.752651095$$
$$P_{93}(-1.01) = -.770177598$$
$$P_{94}(-1.01) = 1.777879382$$
$$P_{95}(-1.01) = -.795658168$$
$$P_{96}(-1.01) = 1.803614758$$
$$P_{97}(-1.01) = -.821650897$$
$$P_{98}(-1.01) = 1.829867414$$
$$P_{99}(-1.01) = -.848166080$$
$$P_{100}(-1.01) = 1.856647749$$

We have no hint of convergence here.

>

>