

Fourier Series

Period 2π over the interval $[-\pi, \pi]$.

```
> restart:with(plots):
```

```
Warning, the name changecoords has been redefined
```

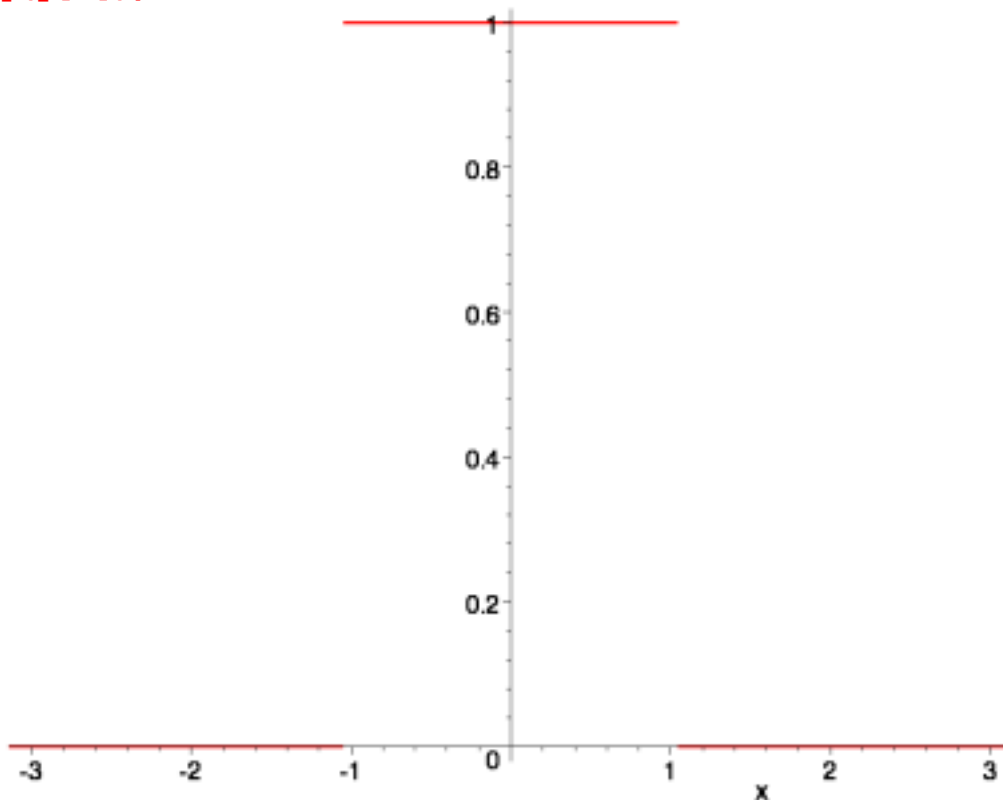
We begin by considering the function

```
> f:=piecewise(x<-Pi/3,0,x<Pi/3,1,0);
```

$$f := \begin{cases} 0 & x < -\frac{1}{3}\pi \\ 1 & x < \frac{1}{3}\pi \\ 0 & \text{otherwise} \end{cases}$$

over the interval $x = -\pi..pi$, whose plot is below.

```
> p[0]:=plot(f,x=-Pi..Pi,thickness=3,discont=true):  
display(p[0]);
```



We will approximate this function by Fourier polynomials of degrees 1 through 15.

```
> n:=15;
```

```
n := 15
```

We first compute the coefficient $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$.

```
> a[0]:=(1/(2*Pi))*int(f,x=-Pi..Pi);
```

$$a_0 := \frac{1}{3}$$

Next we use a loop to compute the coefficients $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$ and $b_k = \frac{1}{\pi}$

$\int_{-\pi}^{\pi} f(x) \sin(kx) dx$, along with the Fourier polynomials

$$F_n(x) = a_0 + \left(\sum_{k=1}^n a_k \cos(kx) \right) + \left(\sum_{k=1}^n b_k \sin(kx) \right) \text{ for } n = 1..15.$$

```
> for k from 1 to n do
  a[k] := (1/Pi)*int(f*cos(k*x), x=-Pi..Pi);
  b[k] := (1/Pi)*int(f*sin(k*x), x=-Pi..Pi);
  F[k] := a[0]+sum('a[i]*cos(i*x)', 'i'=1..k)+sum('b[i]*sin(i*x)', 'i'=1..k
);
od;
```

$$a_1 := \frac{\sqrt{3}}{\pi}$$

$$b_1 := 0$$

$$F_1 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi}$$

$$a_2 := \frac{1}{2} \frac{\sqrt{3}}{\pi}$$

$$b_2 := 0$$

$$F_2 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2} \sqrt{3} \cos(2x)}{\pi}$$

$$a_3 := 0$$

$$b_3 := 0$$

$$F_3 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2} \sqrt{3} \cos(2x)}{\pi}$$

$$a_4 := -\frac{1}{4} \frac{\sqrt{3}}{\pi}$$

$$b_4 := 0$$

$$F_4 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2} \sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi}$$

$$a_5 := -\frac{1}{5} \frac{\sqrt{3}}{\pi}$$

$$b_5 := 0$$

$$F_5 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi}$$

$$a_6 := 0$$

$$b_6 := 0$$

$$F_6 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi}$$

$$a_7 := \frac{1}{7} \frac{\sqrt{3}}{\pi}$$

$$b_7 := 0$$

$$F_7 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$a_8 := \frac{1}{8} \frac{\sqrt{3}}{\pi}$$

$$b_8 := 0$$

$$F_8 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi}$$

$$a_9 := 0$$

$$b_9 := 0$$

$$F_9 := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi}$$

$$a_{10} := -\frac{1}{10} \frac{\sqrt{3}}{\pi}$$

$$b_{10} := 0$$

$$F_{10} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi}$$

$$a_{11} := -\frac{1}{11} \frac{\sqrt{3}}{\pi}$$

$$b_{11} := 0$$

$$F_{11} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi} - \frac{1}{11} \frac{\sqrt{3} \cos(11x)}{\pi}$$

$$a_{12} := 0$$

$$b_{12} := 0$$

$$F_{12} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi} - \frac{1}{11} \frac{\sqrt{3} \cos(11x)}{\pi}$$

$$a_{13} := \frac{1}{13} \frac{\sqrt{3}}{\pi}$$

$$b_{13} := 0$$

$$F_{13} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi} - \frac{1}{11} \frac{\sqrt{3} \cos(11x)}{\pi} + \frac{\frac{1}{13}\sqrt{3} \cos(13x)}{\pi}$$

$$a_{14} := \frac{1}{14} \frac{\sqrt{3}}{\pi}$$

$$b_{14} := 0$$

$$F_{14} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi} - \frac{1}{11} \frac{\sqrt{3} \cos(11x)}{\pi} + \frac{\frac{1}{13}\sqrt{3} \cos(13x)}{\pi}$$

$$+ \frac{\frac{1}{14}\sqrt{3} \cos(14x)}{\pi}$$

$$a_{15} := 0$$

$$b_{15} := 0$$

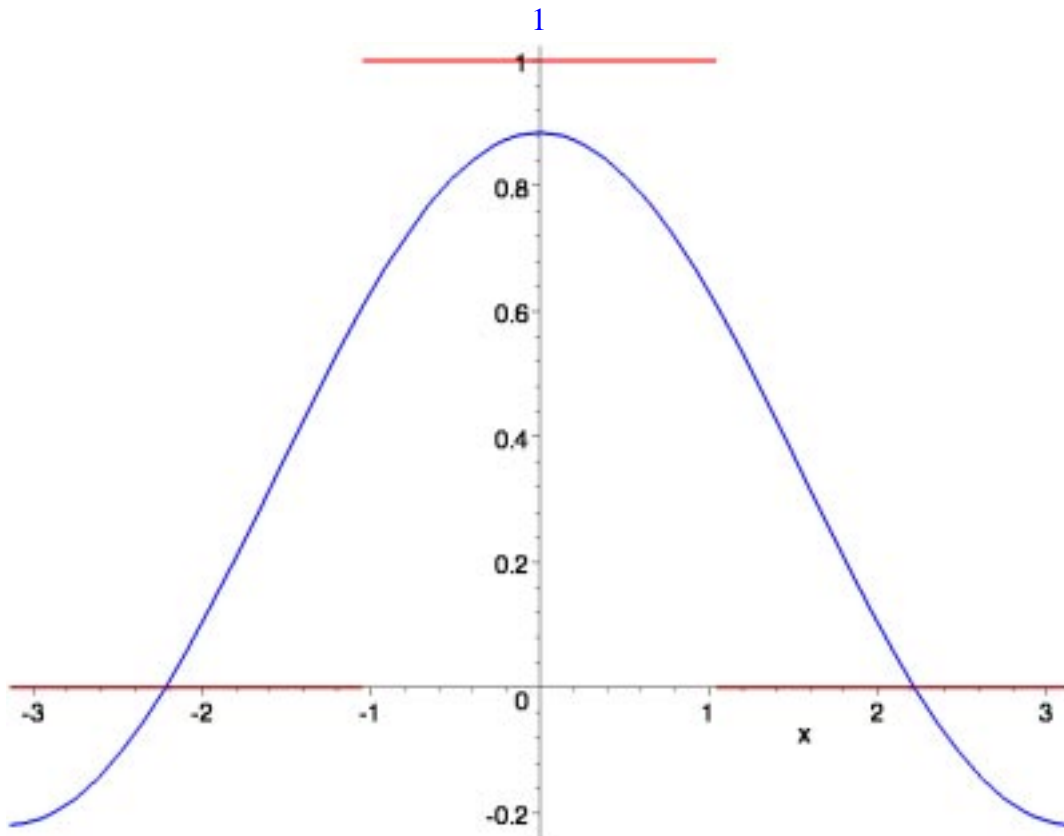
$$F_{15} := \frac{1}{3} + \frac{\sqrt{3} \cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3} \cos(2x)}{\pi} - \frac{1}{4} \frac{\sqrt{3} \cos(4x)}{\pi} - \frac{1}{5} \frac{\sqrt{3} \cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3} \cos(7x)}{\pi}$$

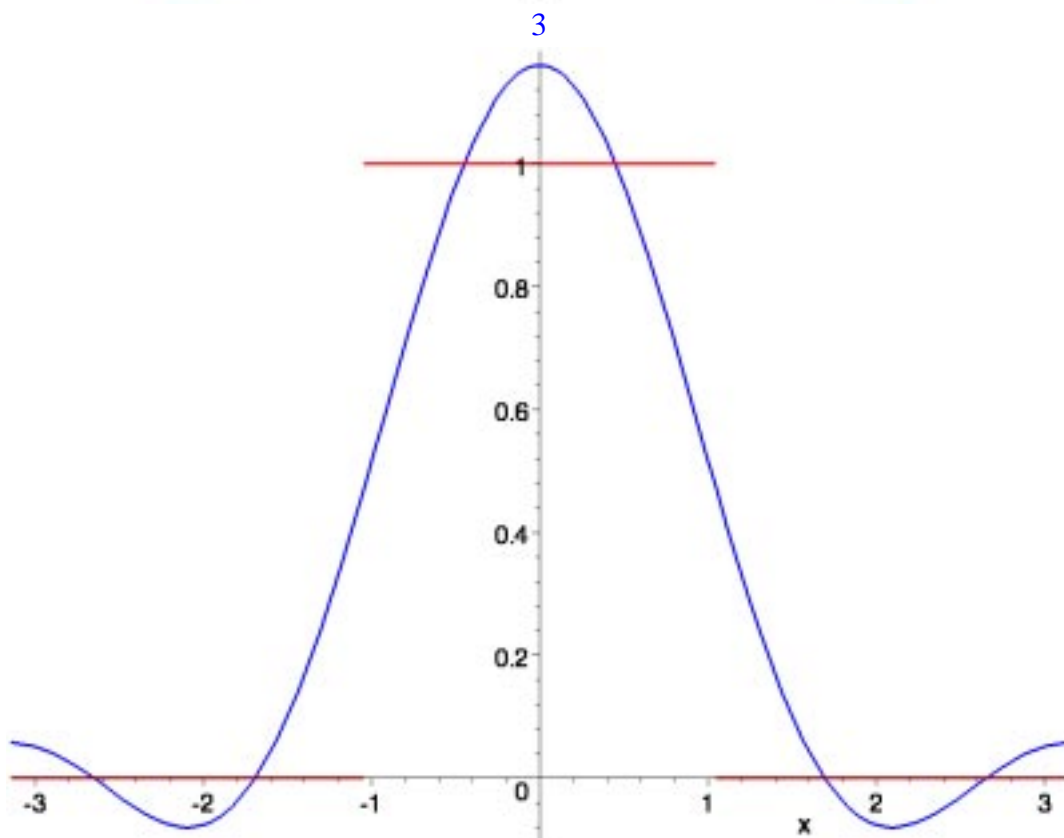
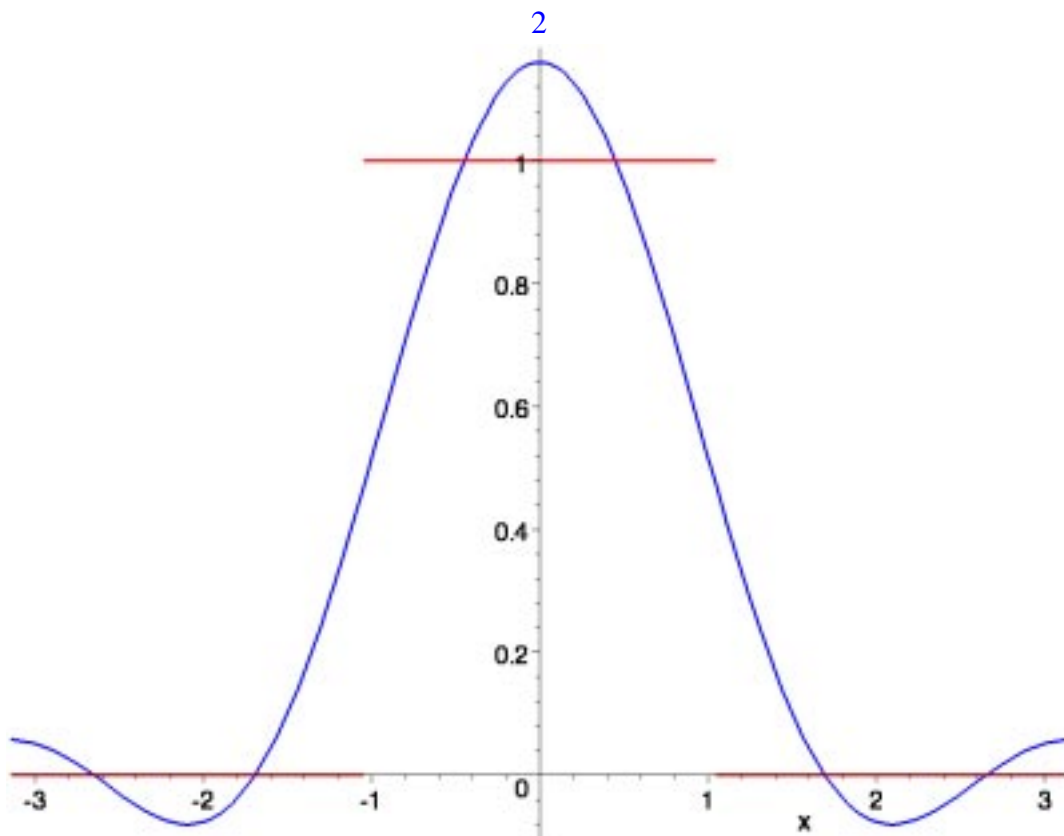
$$+ \frac{\frac{1}{8}\sqrt{3} \cos(8x)}{\pi} - \frac{1}{10} \frac{\sqrt{3} \cos(10x)}{\pi} - \frac{1}{11} \frac{\sqrt{3} \cos(11x)}{\pi} + \frac{\frac{1}{13}\sqrt{3} \cos(13x)}{\pi}$$

$$+ \frac{\frac{1}{14}\sqrt{3} \cos(14x)}{\pi}$$

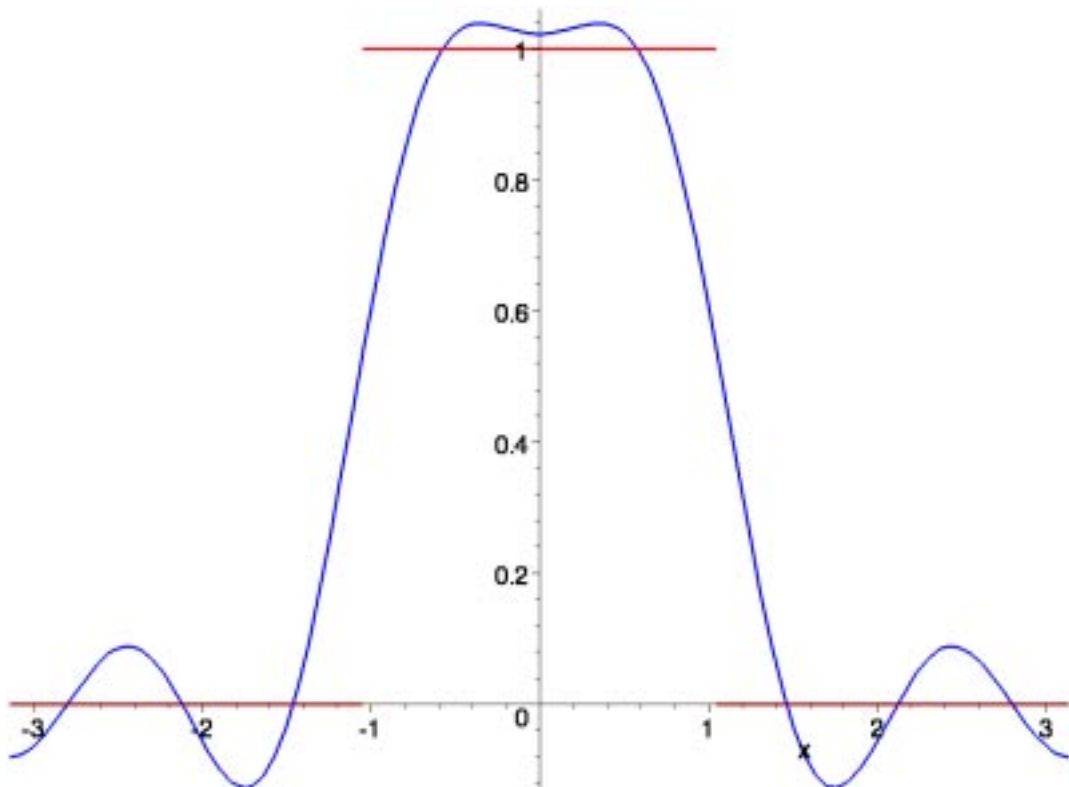
[Next we plot $f(x)$ along with each of the $F_n(x)$ for $n = 1..15$.

```
> for k from 1 to n do
  p[k]:=plot(F[k],x=-Pi..Pi,thickness=3,discont=true,color=blue):
od:
> for k from 1 to n do
  k;
  display(p[0],p[k]):
od;
```

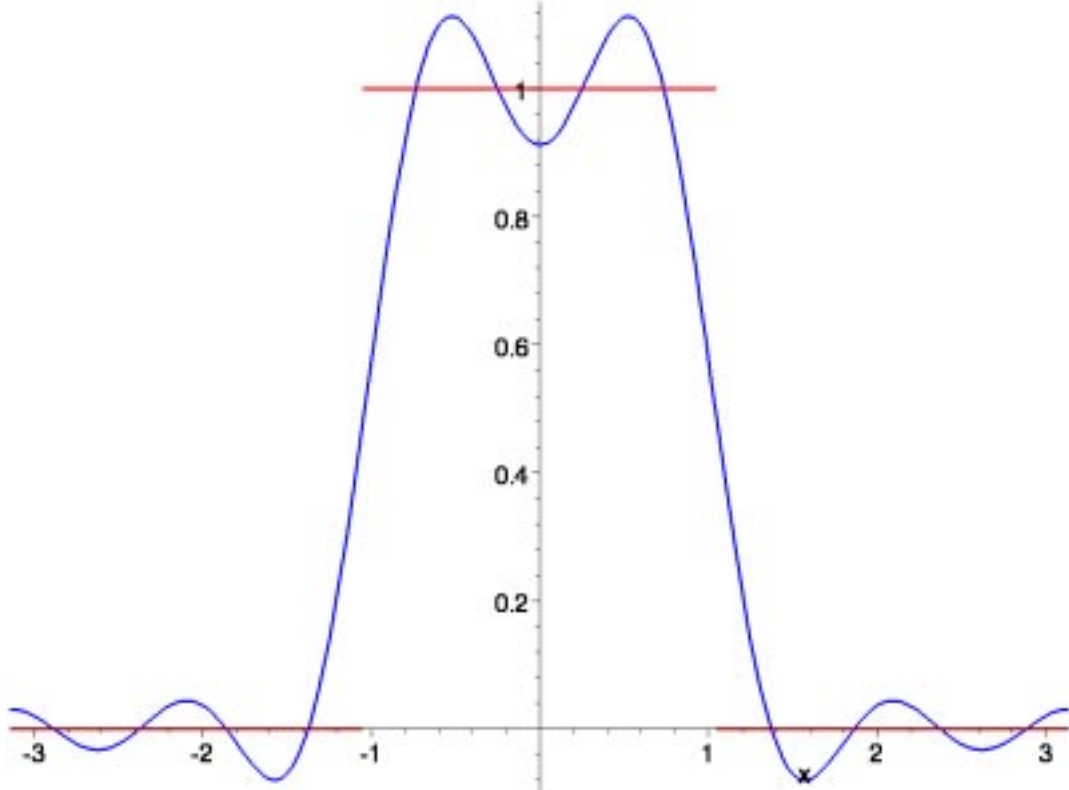




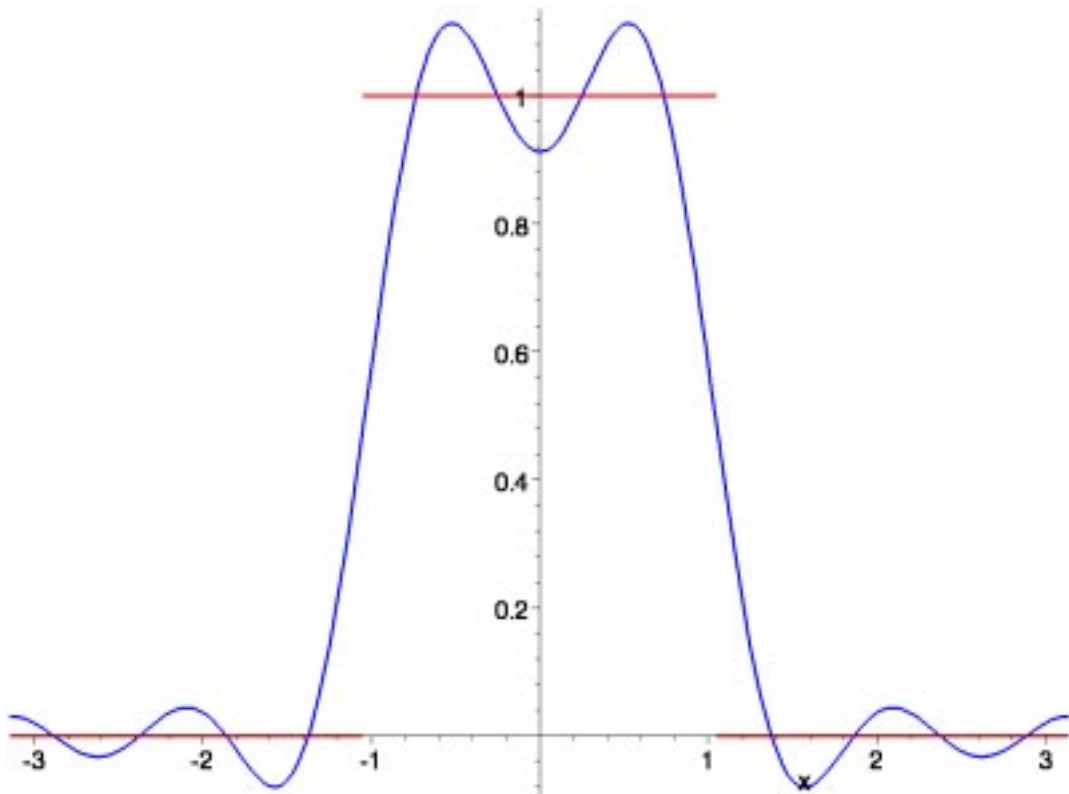
4



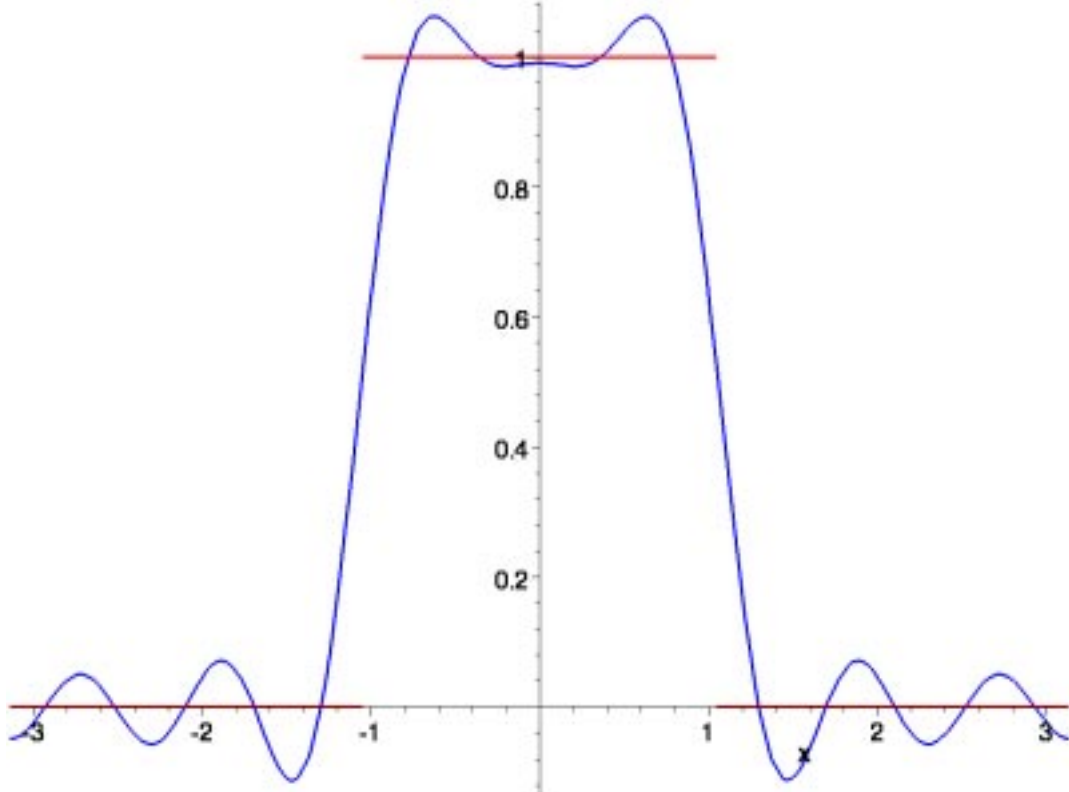
5



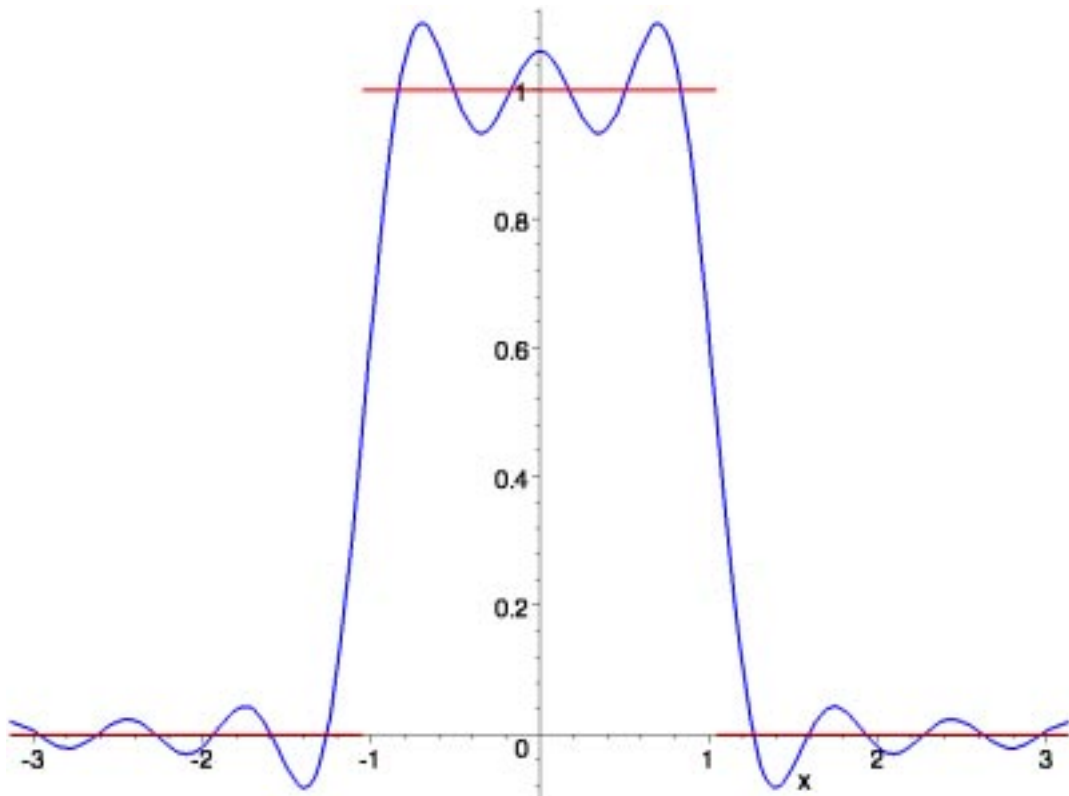
6



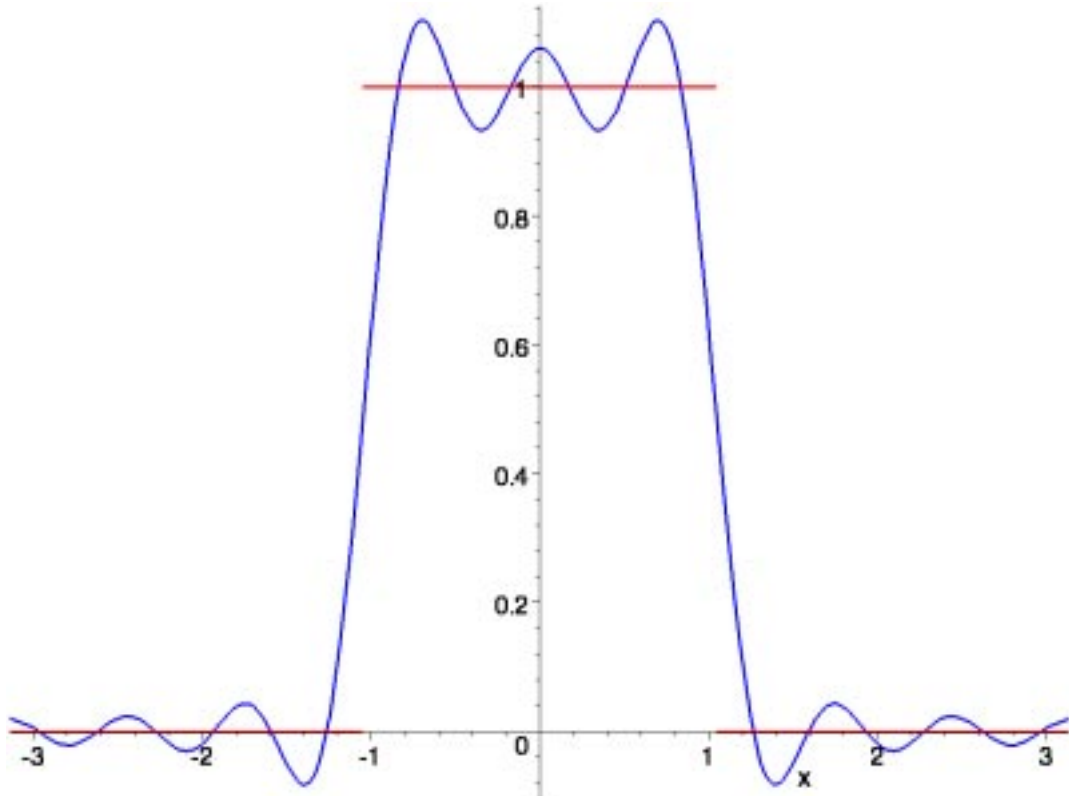
7



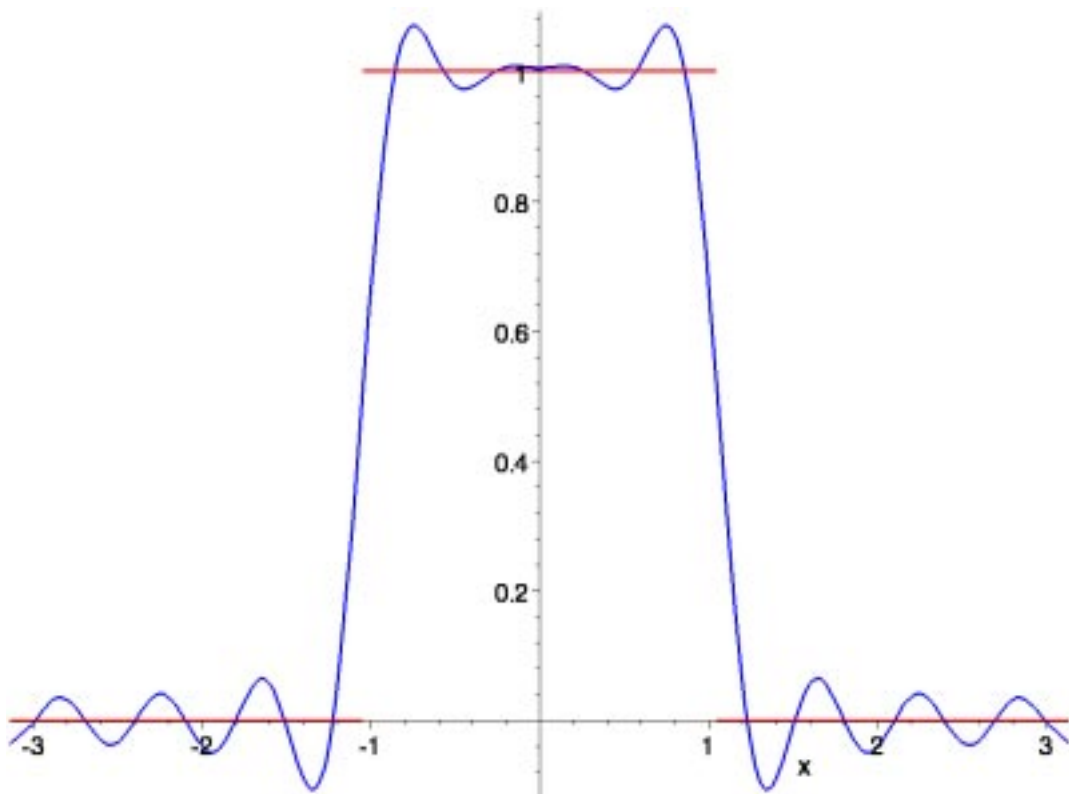
8



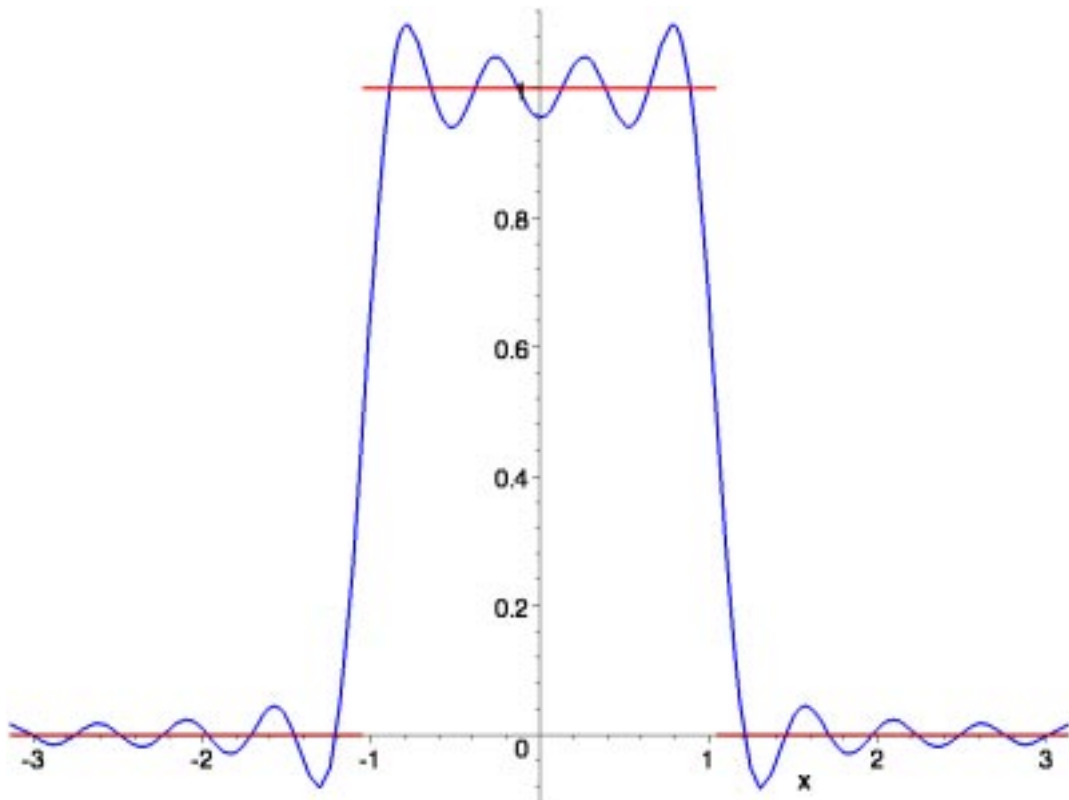
9



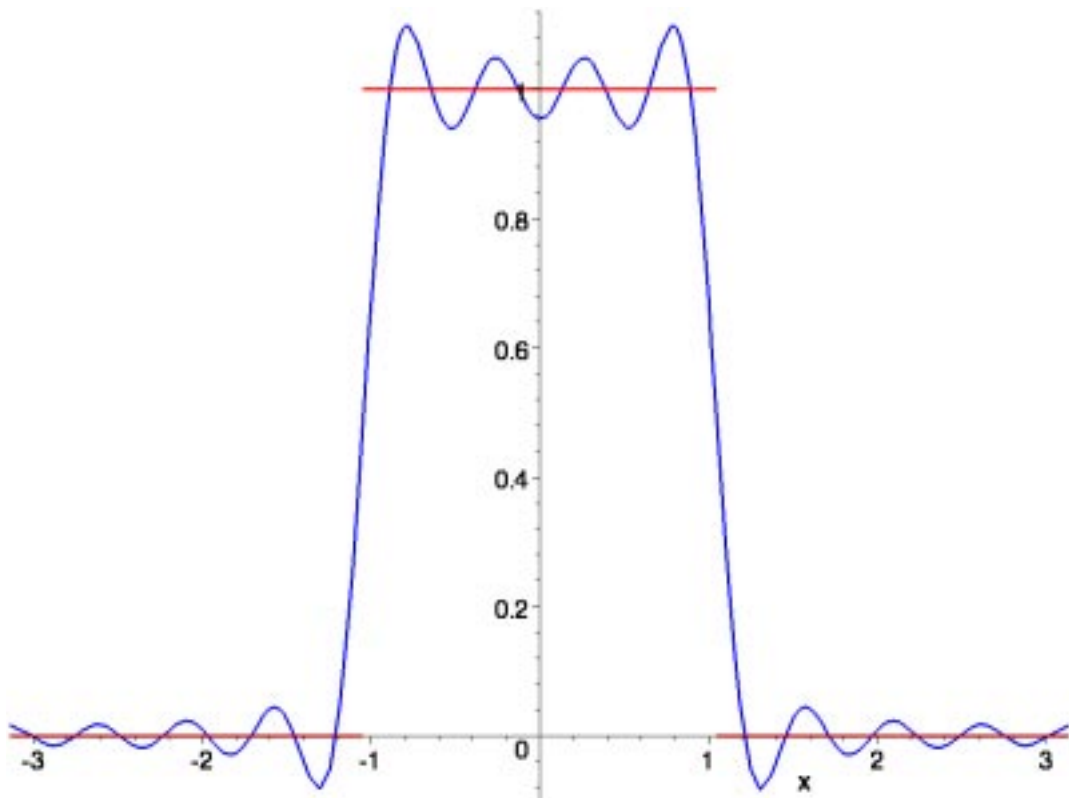
10



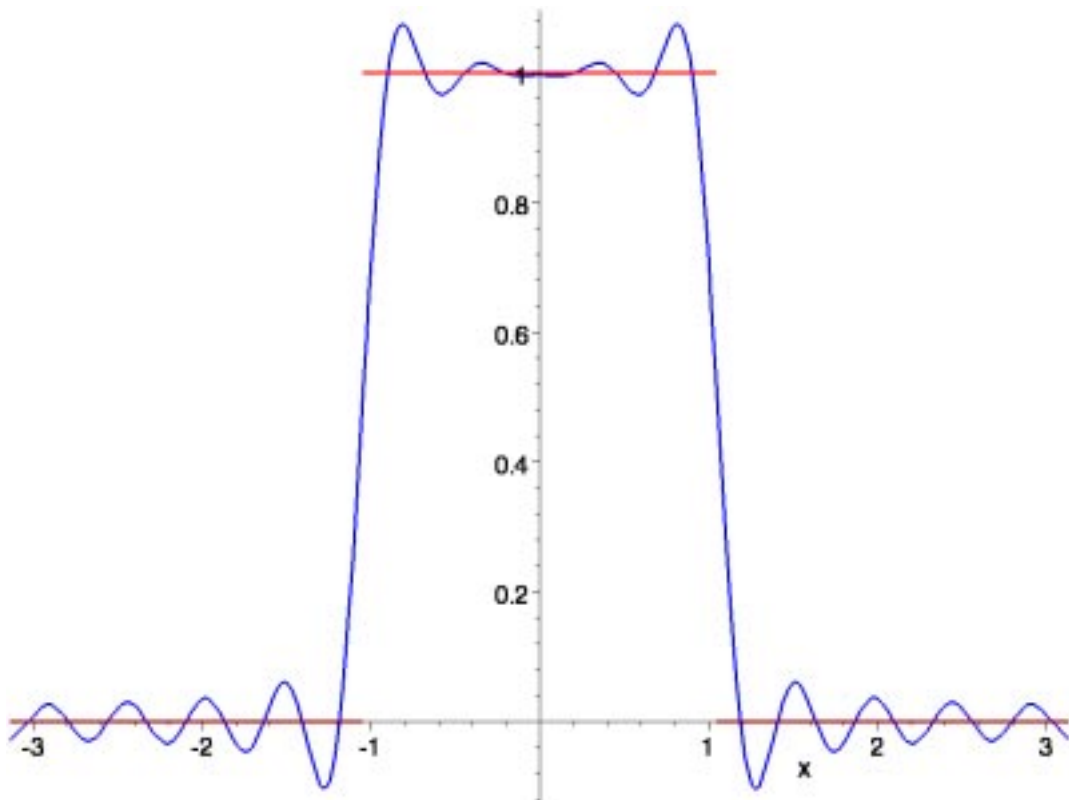
11



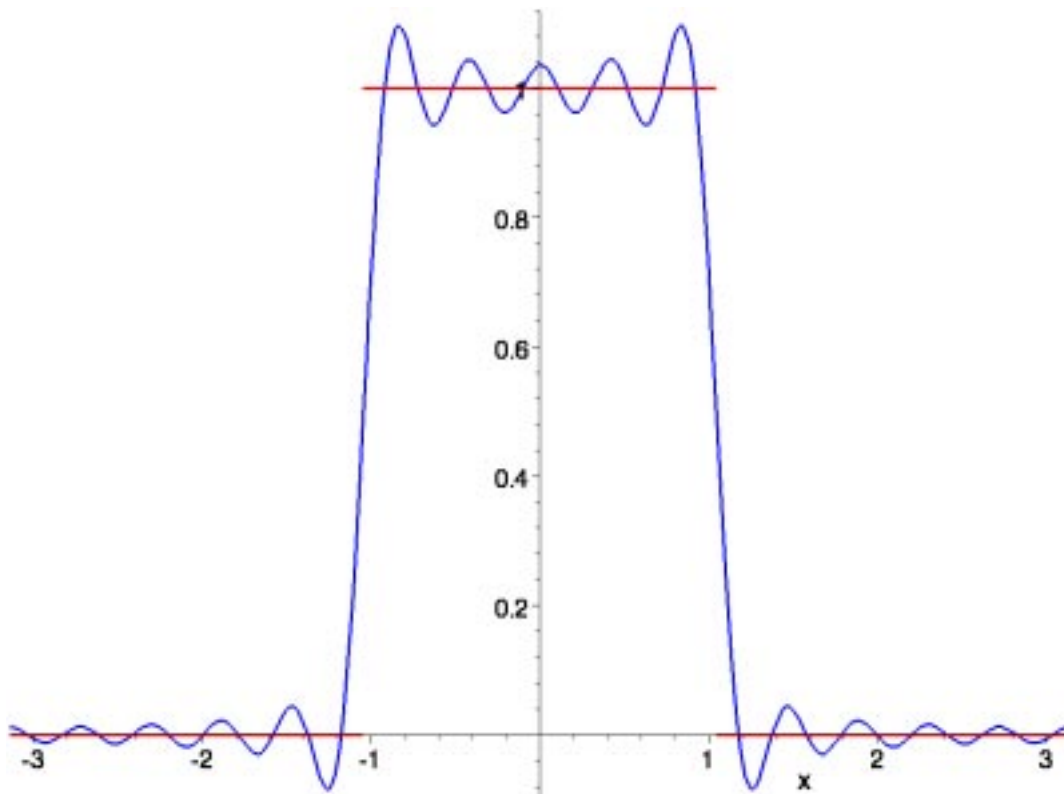
12



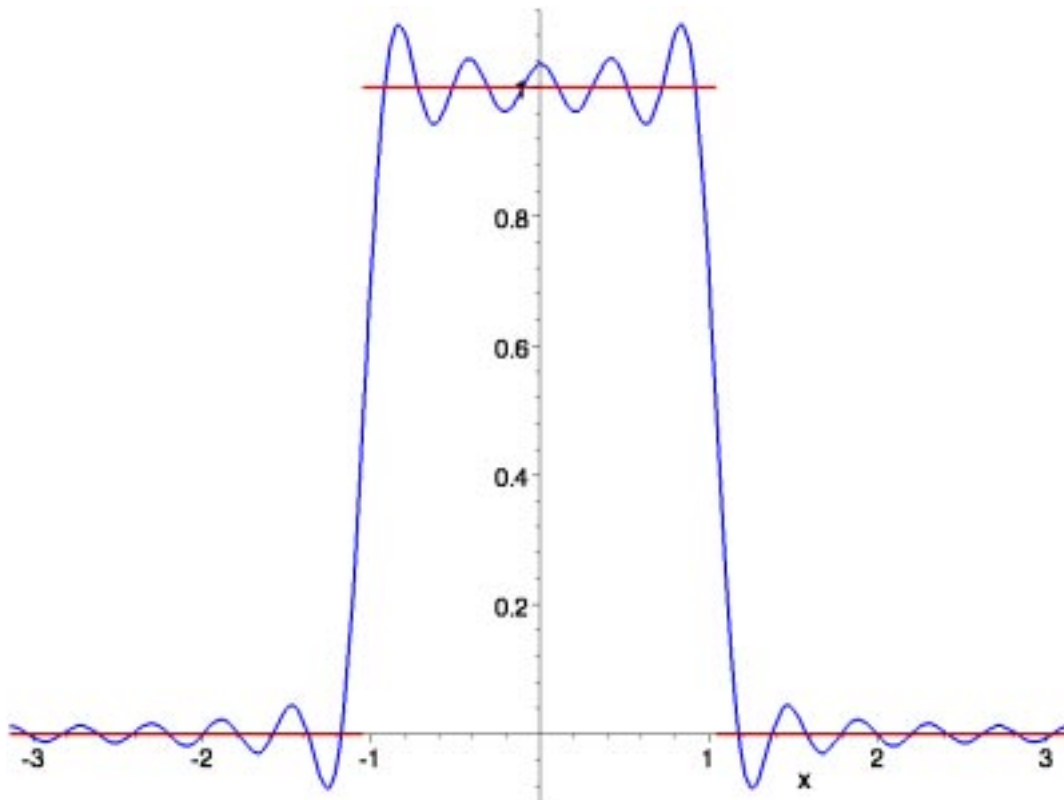
13



14



15



We see that the successive Fourier polynomials give better and better approximations to $f(x)$. Although they may not be as accurate as Taylor polynomials near a given point, they do a better job of approximating a function over an entire interval. They are especially good for approximating

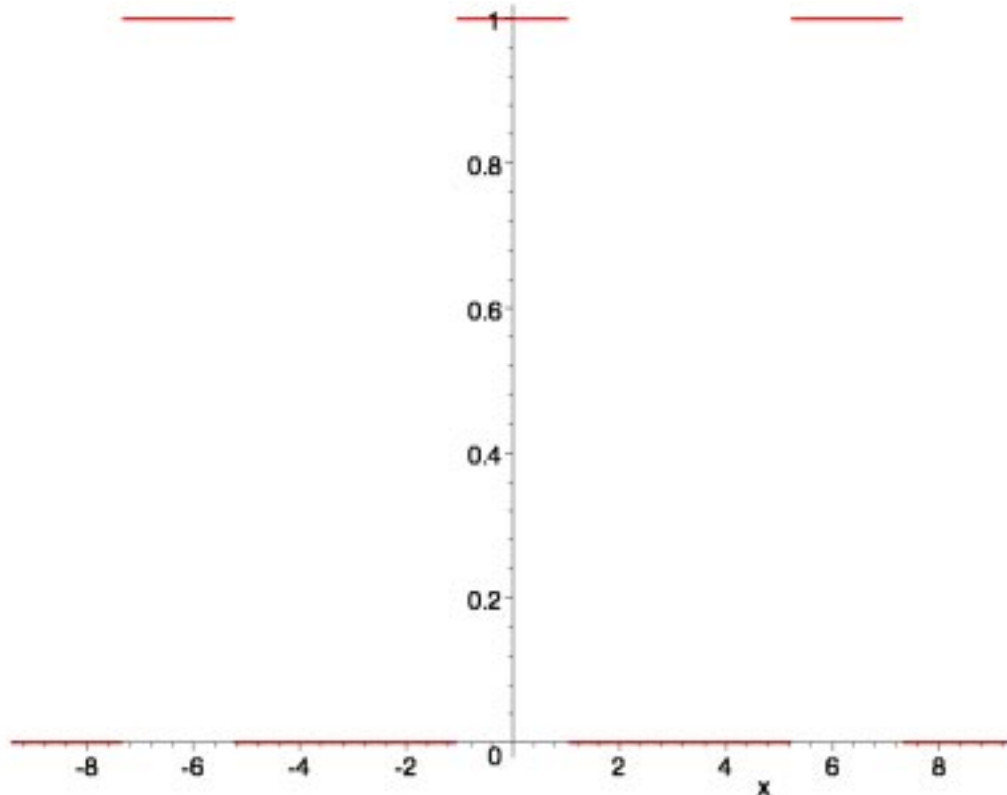
periodic functions. Let's extend $f(x)$ by a period in each direction with $x = -3\pi..3\pi$.

```
> f:=piecewise(x<-7*Pi/3,0,x<-5*Pi/3,1,x<-Pi/3,0,x<Pi/3,1,x<5*Pi/3,0,x<7*Pi/3,1,0);
```

$$f := \begin{cases} 0 & x < -\frac{7}{3}\pi \\ 1 & x < -\frac{5}{3}\pi \\ 0 & x < -\frac{1}{3}\pi \\ 1 & x < \frac{1}{3}\pi \\ 0 & x < \frac{5}{3}\pi \\ 1 & x < \frac{7}{3}\pi \\ 0 & \text{otherwise} \end{cases}$$

We graph the extension.

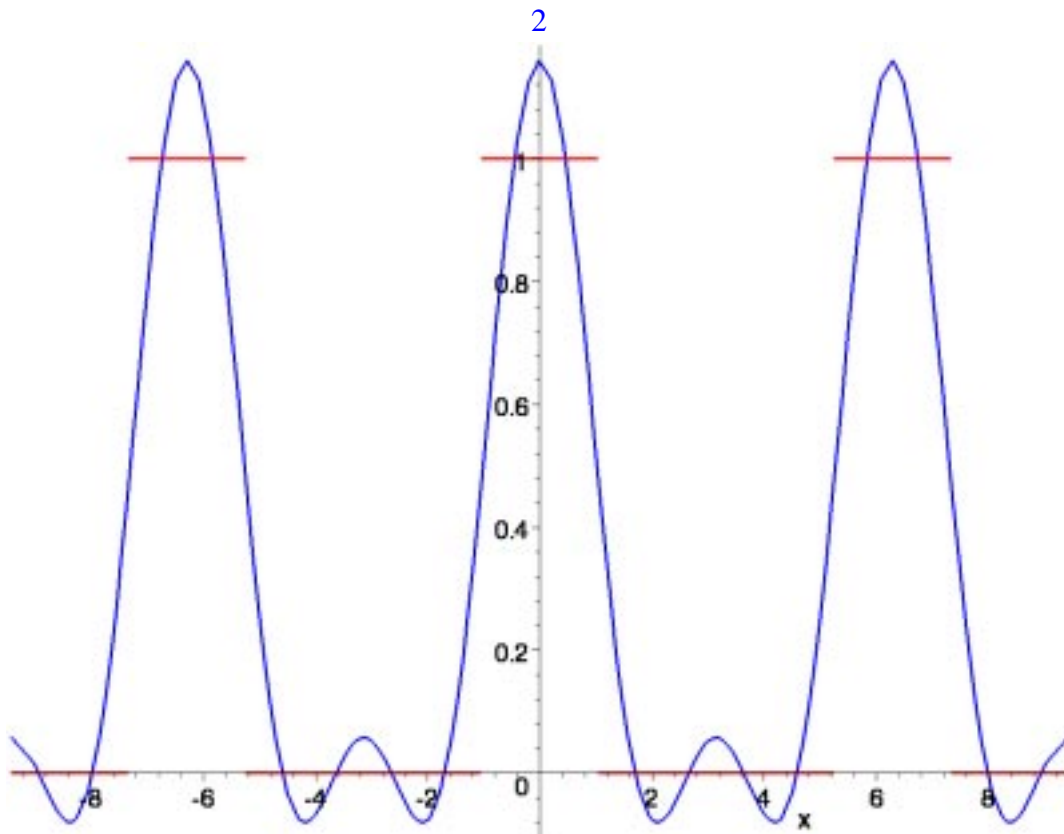
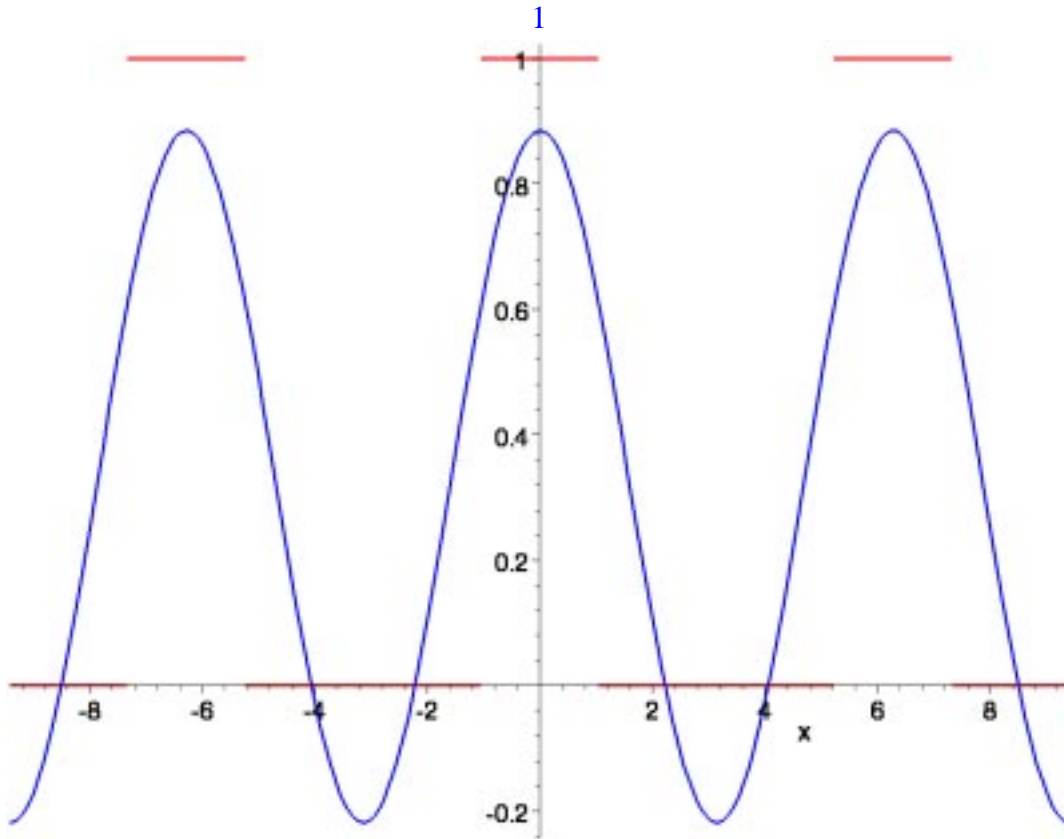
```
> p[0]:=plot(f,x=-3*Pi..3*Pi,thickness=3,discont=true):  
display(p[0]);
```



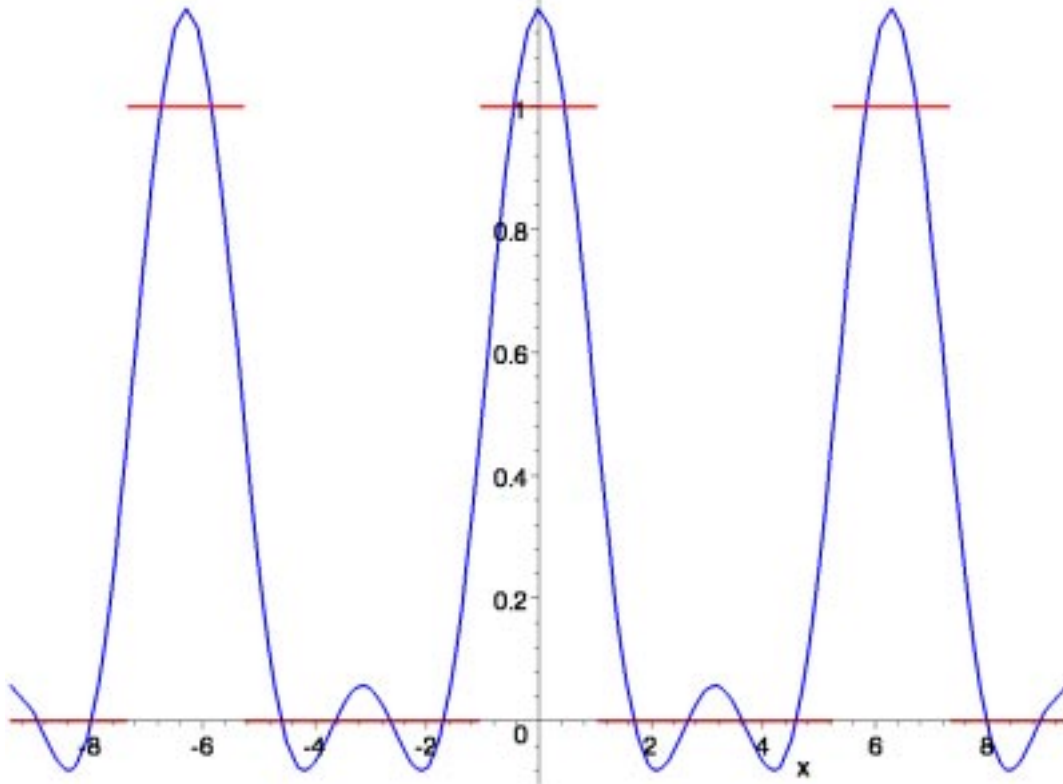
We now plot the extended $f(x)$ along with each of the $F_n(x)$ for $n = 1..15$.

```
> for k from 1 to n do  
  p[k]:=plot(F[k],x=-3*Pi..3*Pi,thickness=3,discont=true,color=blue):  
  od:  
> for k from 1 to n do  
  k;
```

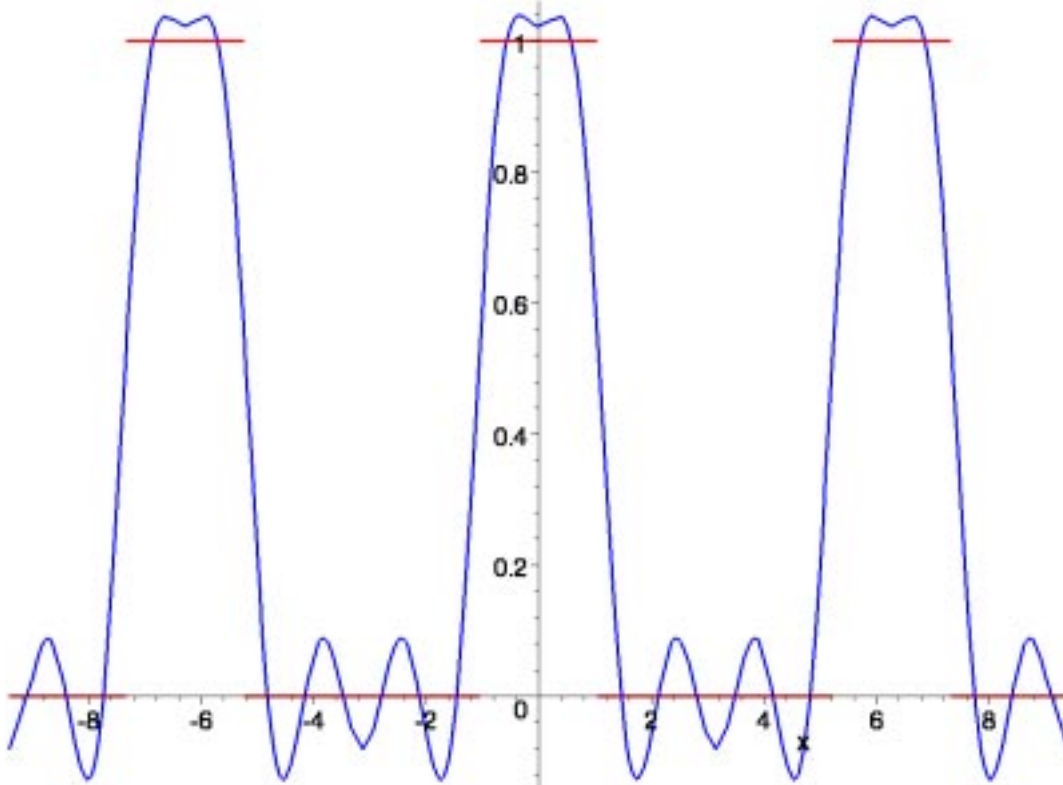
```
display(p[0],p[k]):  
od;
```



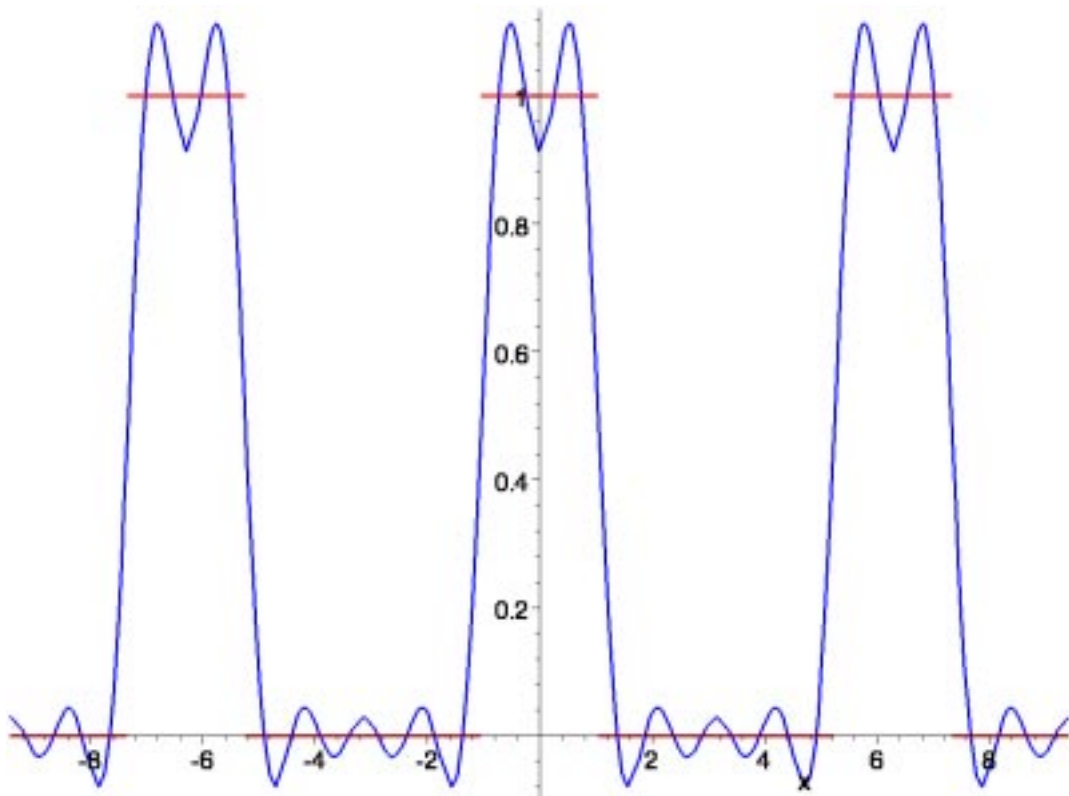
3



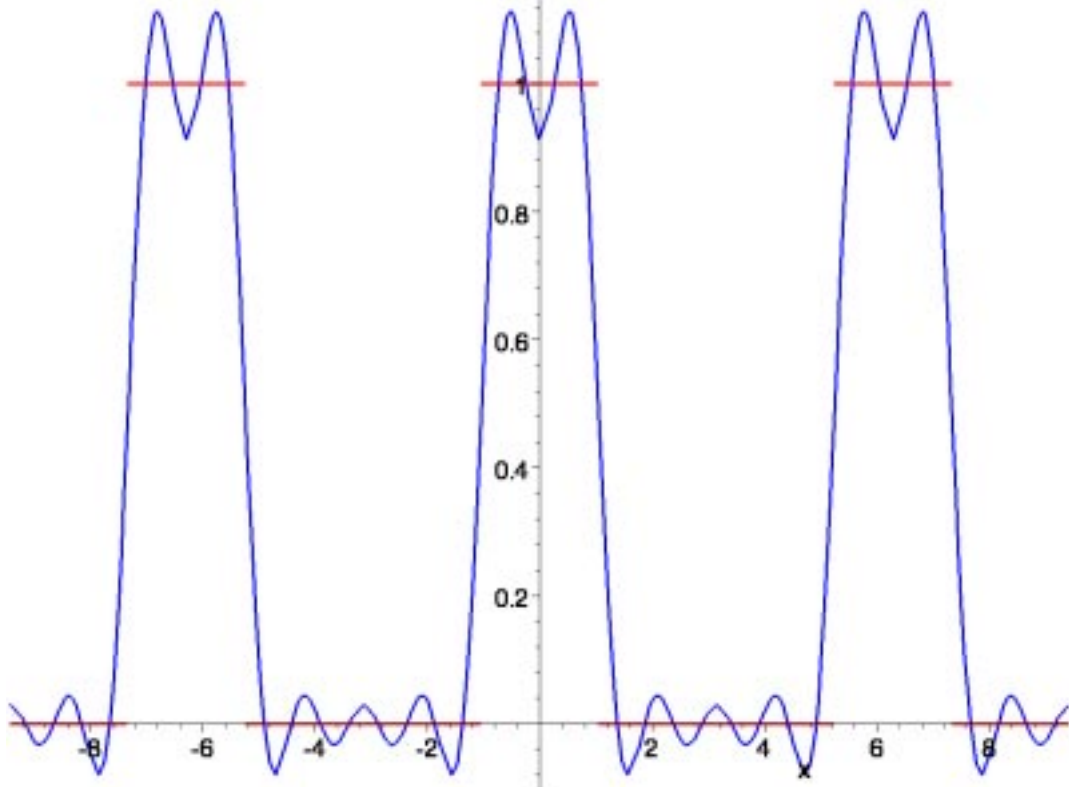
4



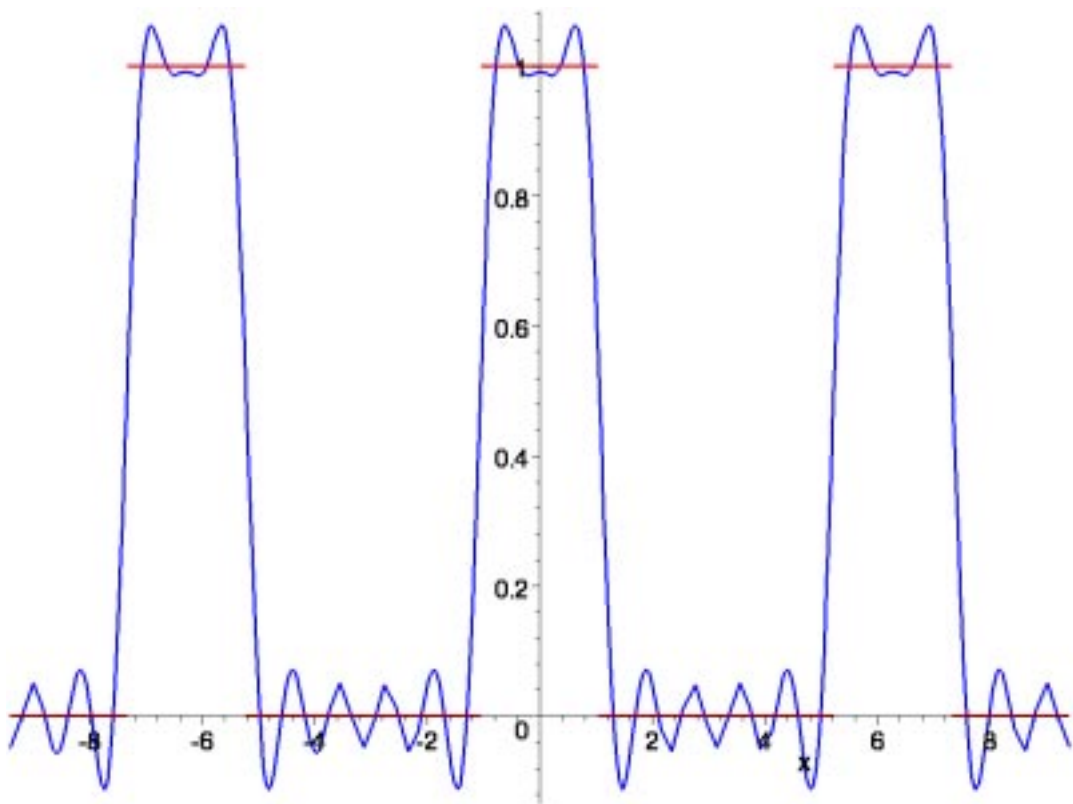
5



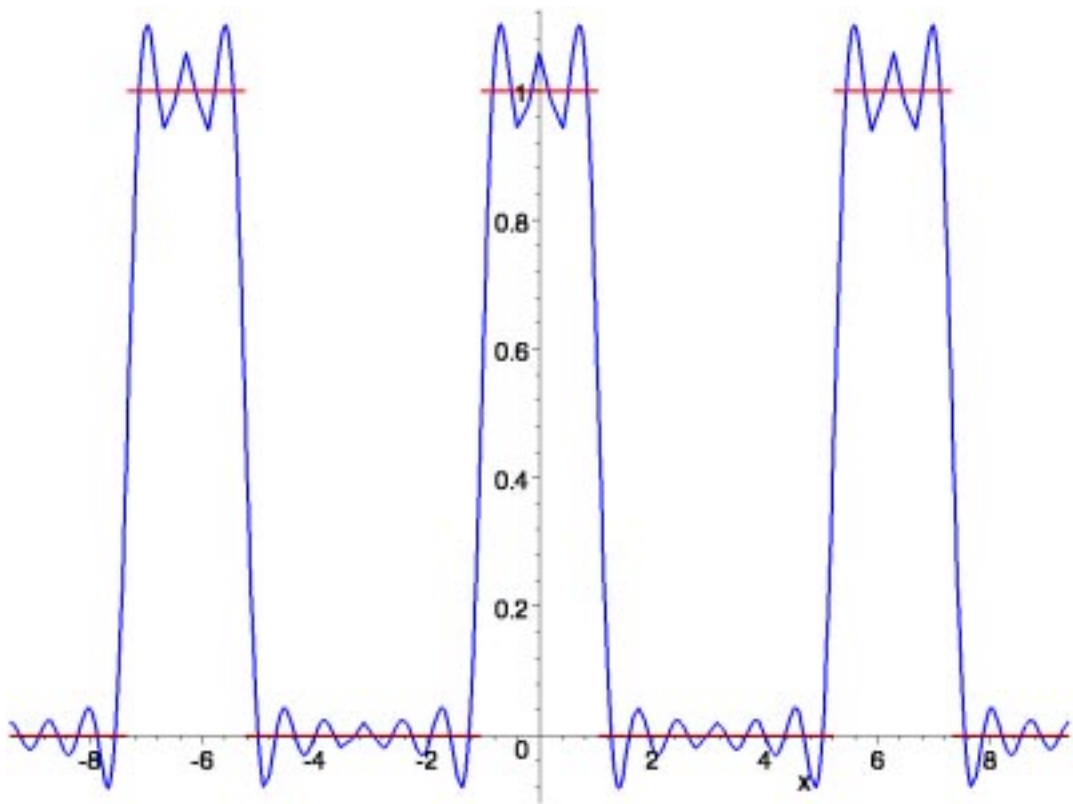
6



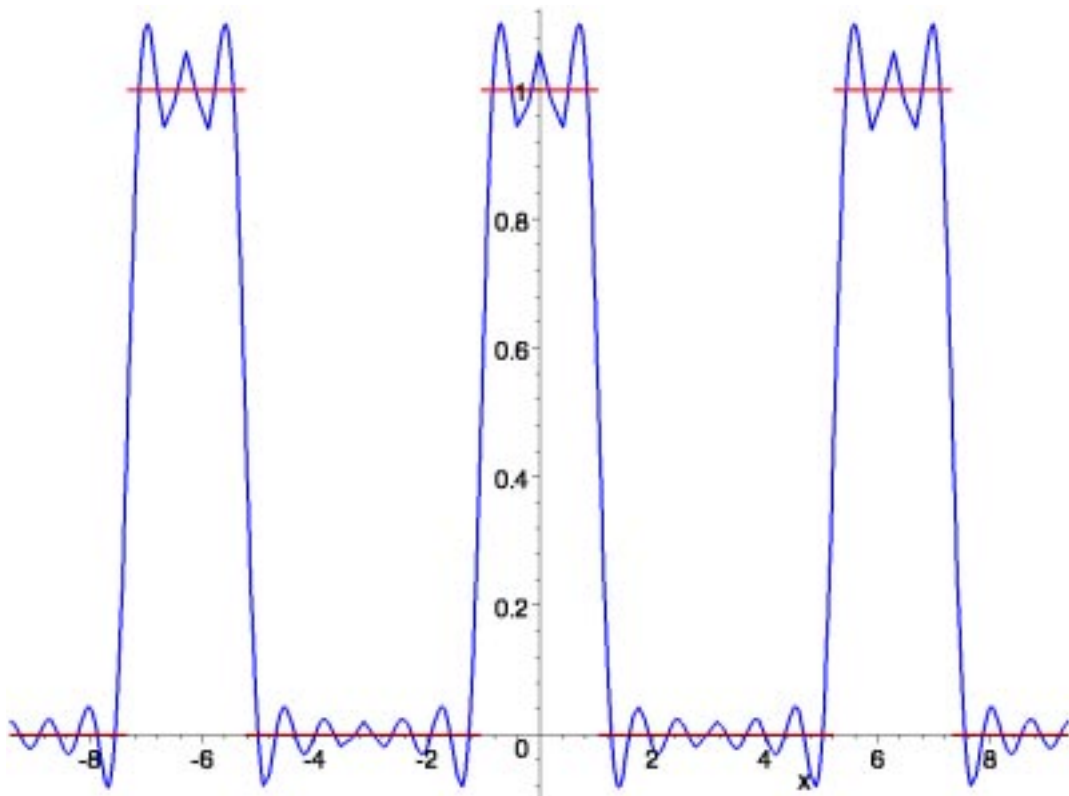
7



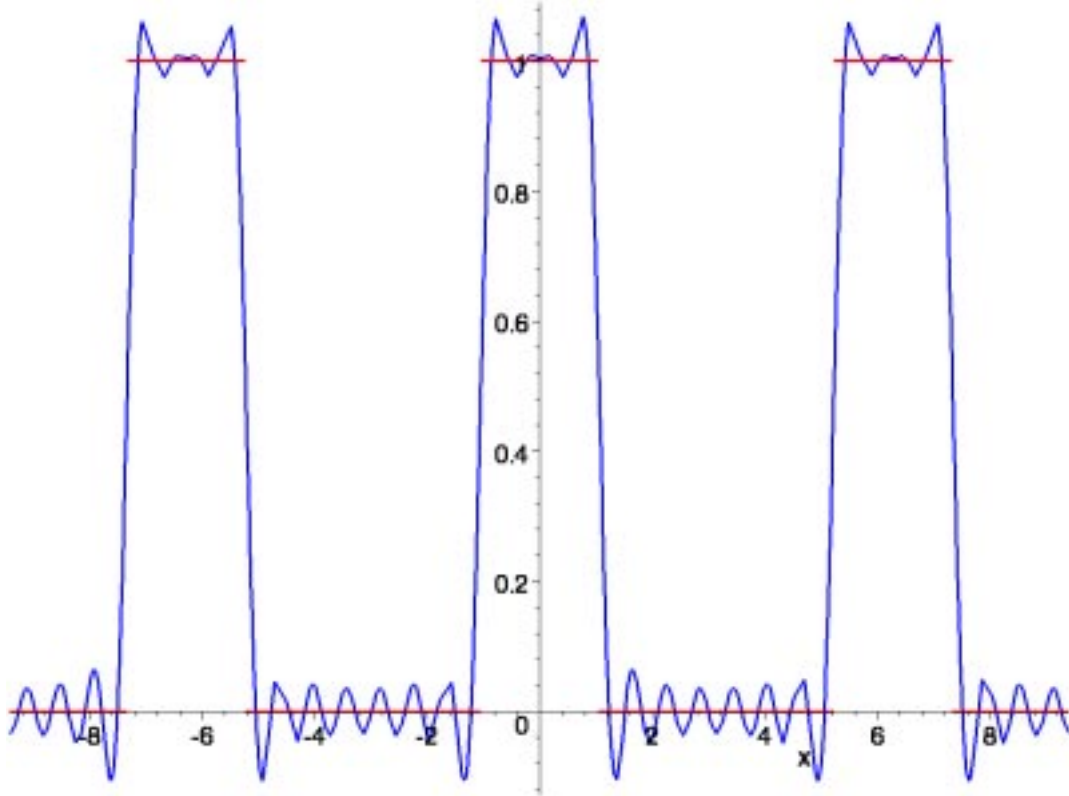
8



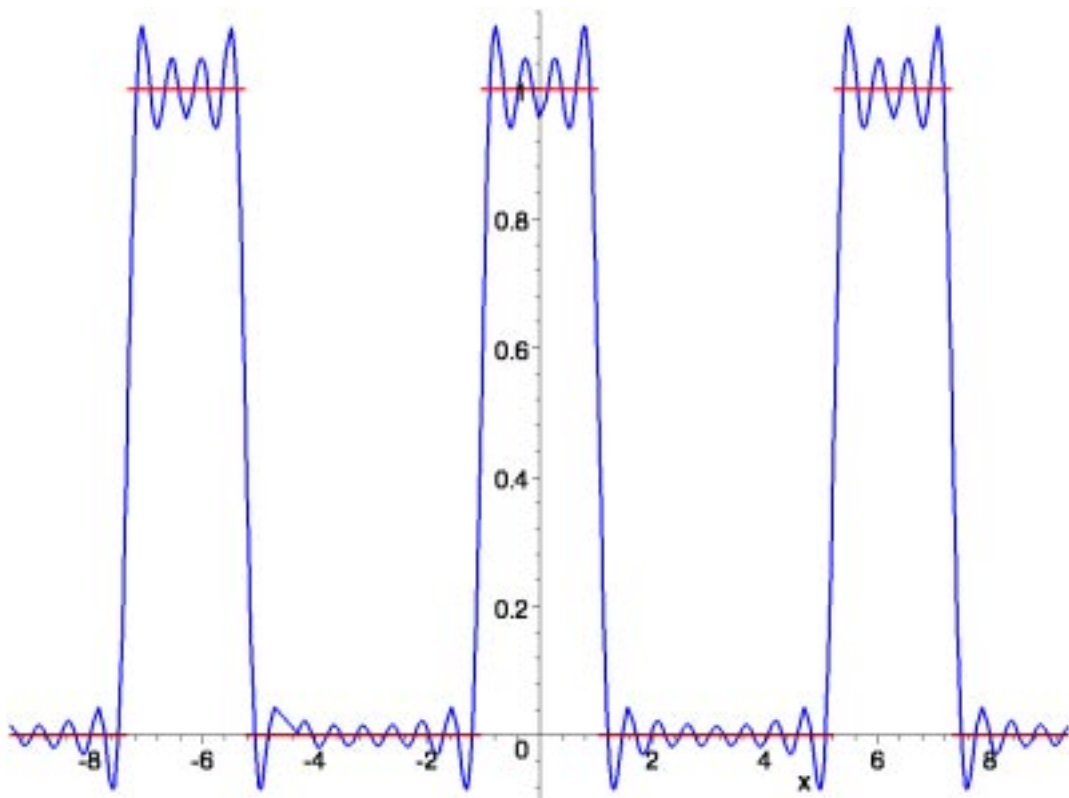
9



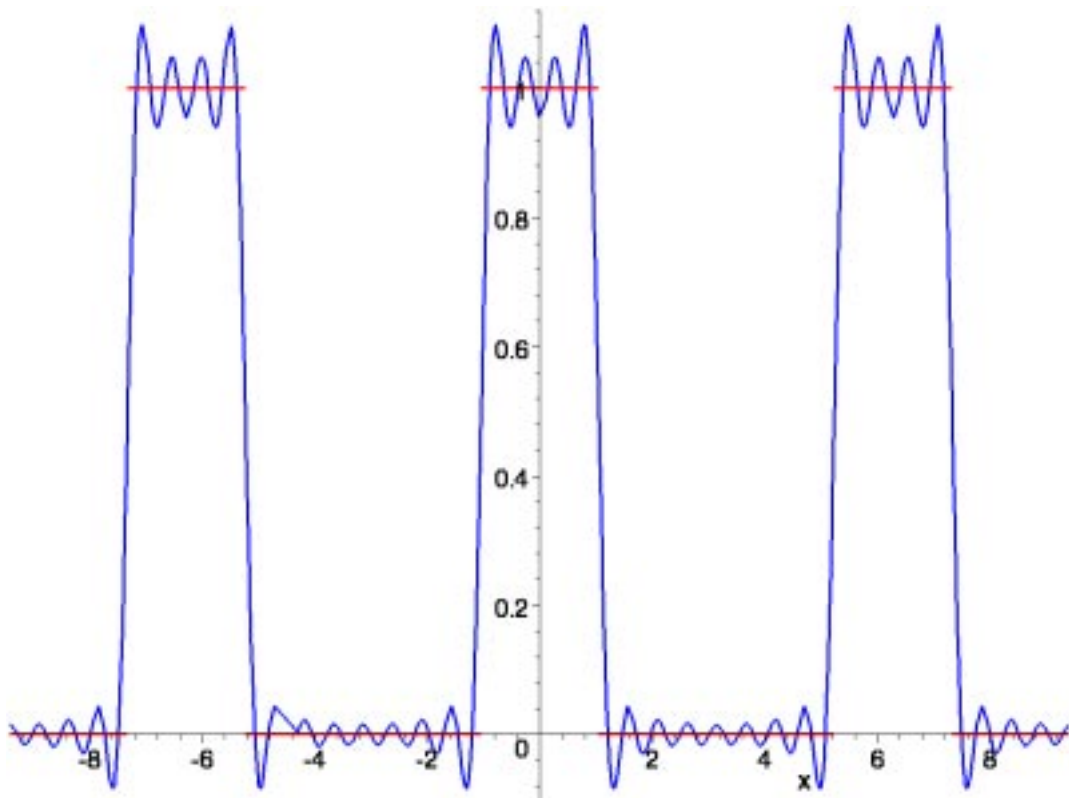
10



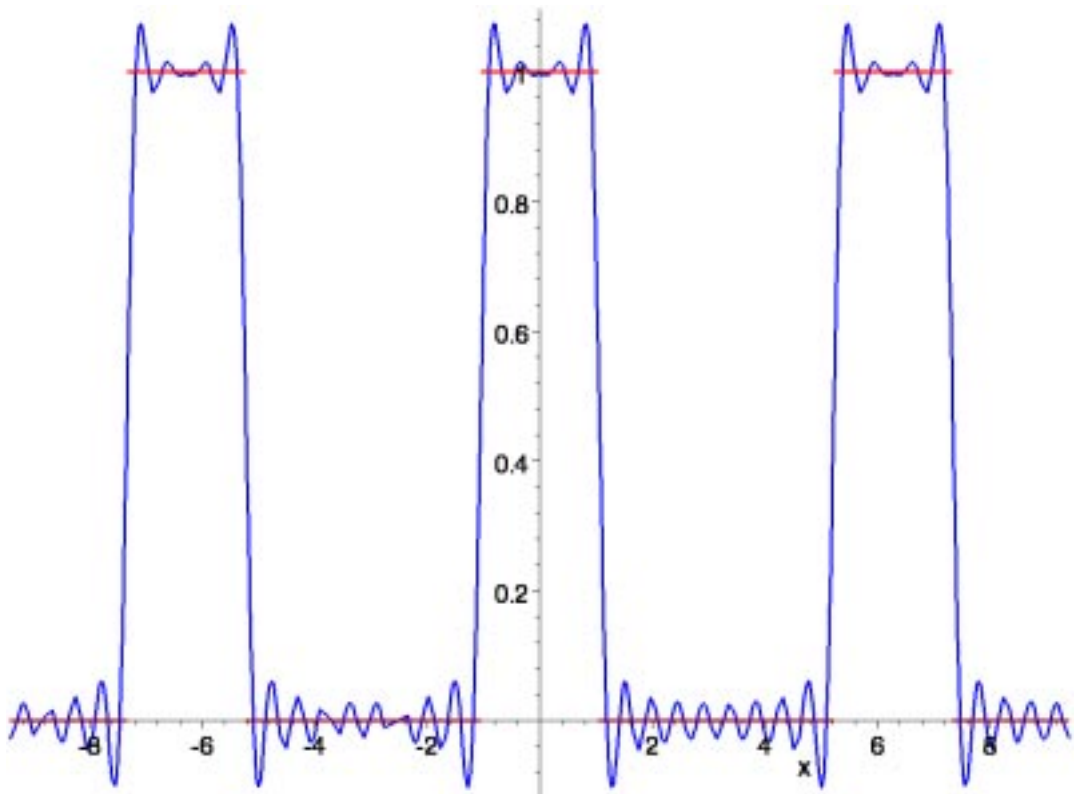
11



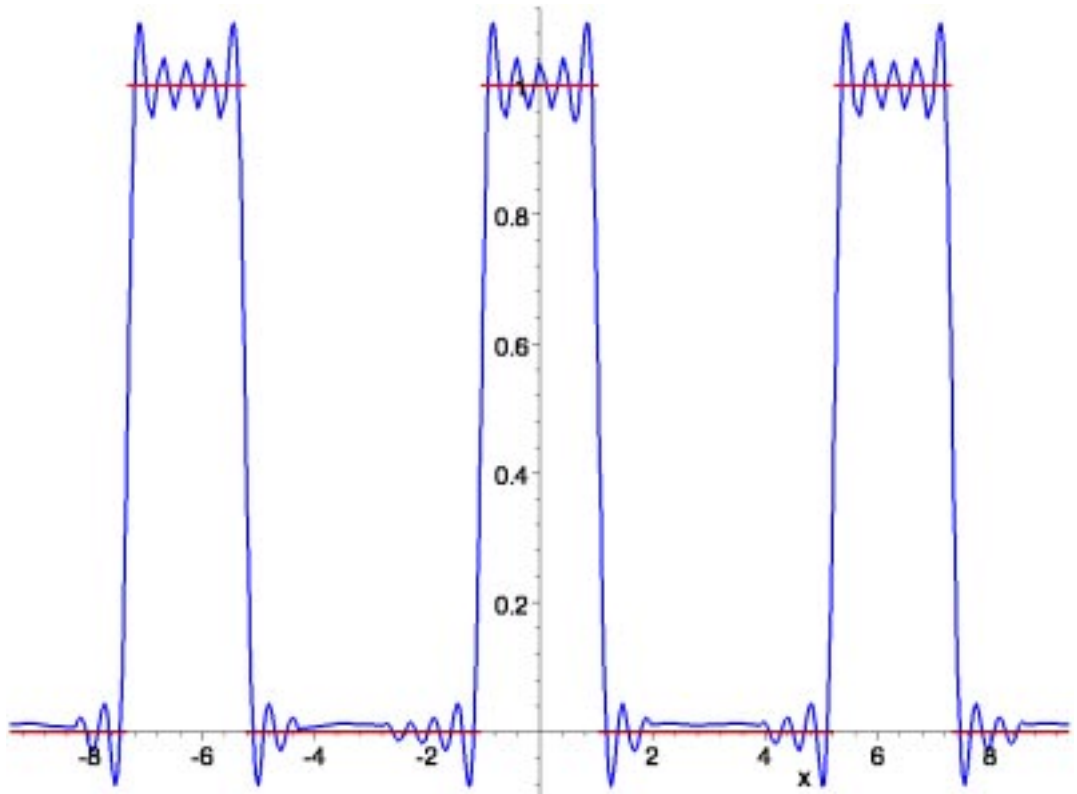
12



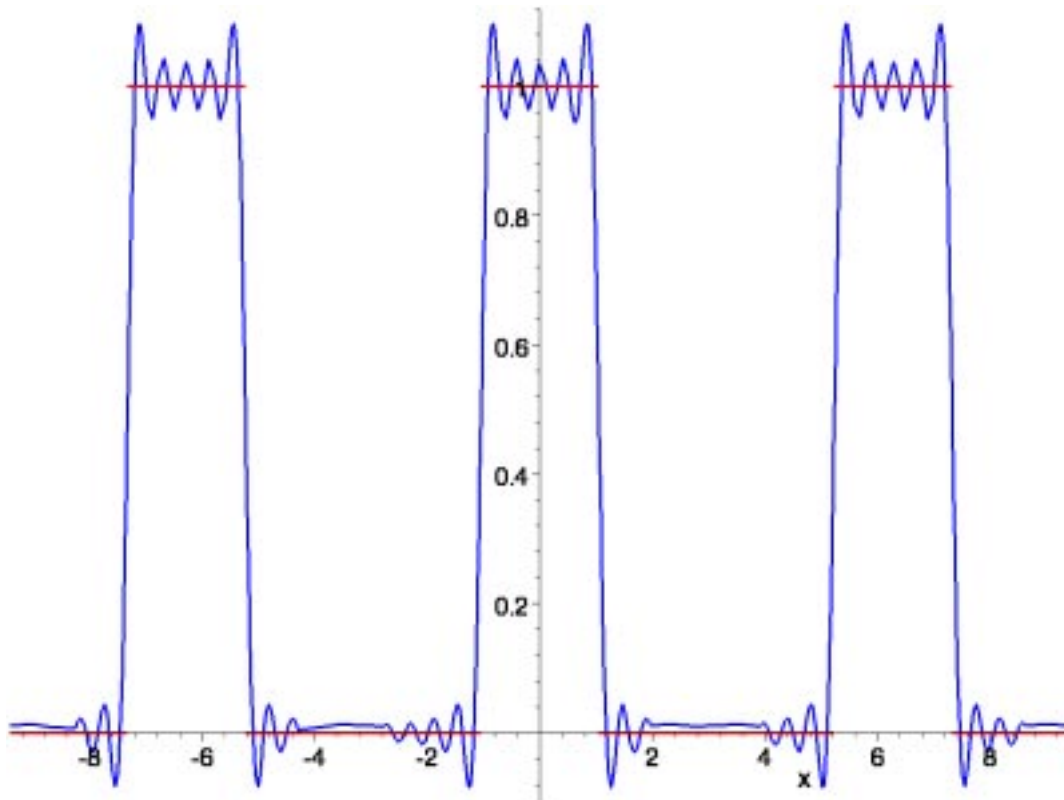
13



14



15



>

Period b over the interval $[-\frac{b}{2}, \frac{b}{2}]$.

>

```
> restart:with(plots):
Warning, the name changecoords has been redefined
```

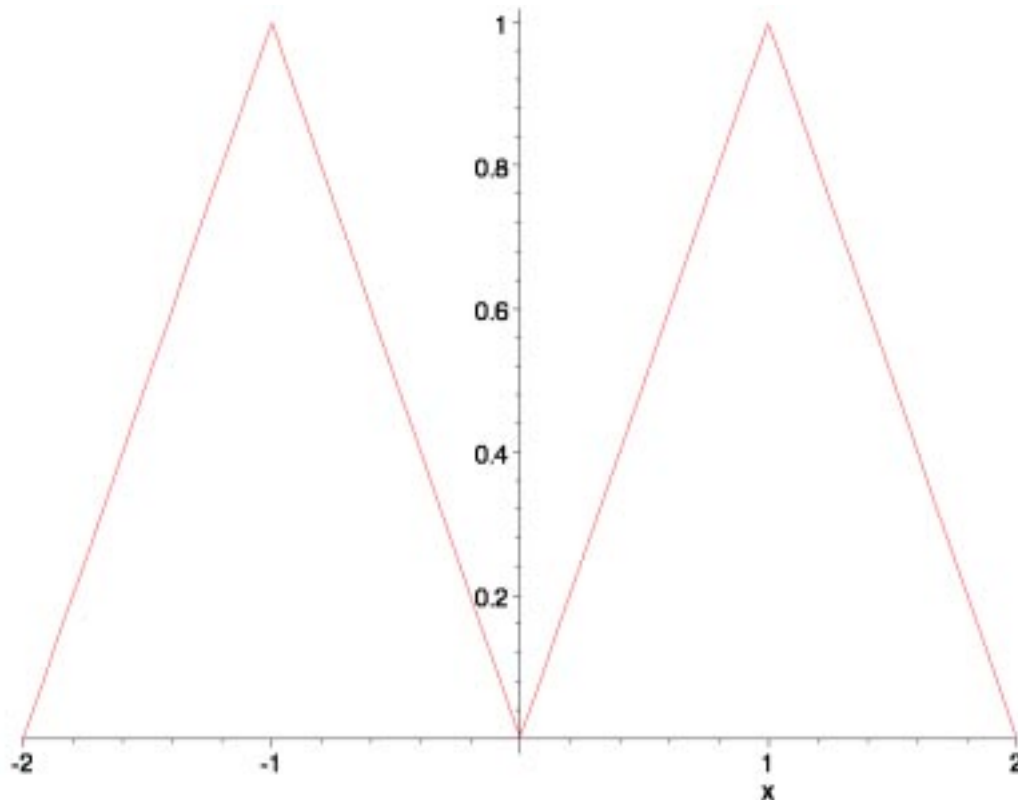
We begin by considering the triangular wave function

```
> f:=piecewise(x<-2,0,x<-1,x+2,x<0,-x,x<1,x,x<2,2-x,0);
```

$$f := \begin{cases} 0 & x < -2 \\ x + 2 & x < -1 \\ -x & x < 0 \\ x & x < 1 \\ 2 - x & x < 2 \\ 0 & \text{otherwise} \end{cases}$$

over the interval $x = -2..2$ with period $b = 2$, whose plot is below.

```
> p[0]:=plot(f,x=-2..2):
display(p[0]);
```



We will approximate this function by Fourier polynomials of degrees 1 through 15.

> `n:=15;`

`n := 15`

We first compute the coefficient $a_0 = \frac{1}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} f(x) dx$.

> `a[0]:=(1/2)*int(f,x=-1..1);`

`a0 := 1/2`

Next we use a loop to compute the coefficients $a_k = \frac{2}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} f(x) \cos\left(\frac{2\pi kx}{b}\right) dx$ and $b_k = \frac{2}{b}$

$\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x) \sin\left(\frac{2\pi kx}{b}\right) dx$, along with the Fourier polynomials

$F_n(x) = a_0 + \left(\sum_{k=1}^n a_k \cos\left(\frac{2\pi kx}{b}\right) \right) + \left(\sum_{k=1}^n b_k \sin\left(\frac{2\pi kx}{b}\right) \right)$ for $n = 1..15$, keeping in mind that $b = 2$

here.

```

> for k from 1 to n do
a[k]:=int(f*cos(Pi*k*x),x=-1..1);
b[k]:=int(f*sin(Pi*k*x),x=-1..1);
F[k]:=a[0]+sum('a[i]*cos(Pi*i*x)', 'i'=1..k)+sum('b[i]*sin(Pi*i*x)', 'i'=1..k);
od;

```

$$a_1 := -4 \frac{1}{\pi^2}$$

$$b_1 := 0$$

$$F_1 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2}$$

$$a_2 := 0$$

$$b_2 := 0$$

$$F_2 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2}$$

$$a_3 := -\frac{4}{9} \frac{1}{\pi^2}$$

$$b_3 := 0$$

$$F_3 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2}$$

$$a_4 := 0$$

$$b_4 := 0$$

$$F_4 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2}$$

$$a_5 := -\frac{4}{25} \frac{1}{\pi^2}$$

$$b_5 := 0$$

$$F_5 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2}$$

$$a_6 := 0$$

$$b_6 := 0$$

$$F_6 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2}$$

$$a_7 := -\frac{4}{49} \frac{1}{\pi^2}$$

$$b_7 := 0$$

$$F_7 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2}$$

$$a_8 := 0$$

$$b_8 := 0$$

$$F_8 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2}$$

$$a_9 := -\frac{4}{81} \frac{1}{\pi^2}$$

$$b_9 := 0$$

$$F_9 := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2}$$

$$a_{10} := 0$$

$$b_{10} := 0$$

$$F_{10} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2}$$

$$a_{11} := -\frac{4}{121} \frac{1}{\pi^2}$$

$$b_{11} := 0$$

$$F_{11} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2} - \frac{4 \cos(11 \pi x)}{121 \pi^2}$$

$$a_{12} := 0$$

$$b_{12} := 0$$

$$F_{12} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2} - \frac{4 \cos(11 \pi x)}{121 \pi^2}$$

$$a_{13} := -\frac{4}{169} \frac{1}{\pi^2}$$

$$b_{13} := 0$$

$$F_{13} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2} - \frac{4 \cos(11 \pi x)}{121 \pi^2} - \frac{4 \cos(13 \pi x)}{169 \pi^2}$$

$$a_{14} := 0$$

$$b_{14} := 0$$

$$F_{14} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2} - \frac{4 \cos(11 \pi x)}{121 \pi^2} - \frac{4 \cos(13 \pi x)}{169 \pi^2}$$

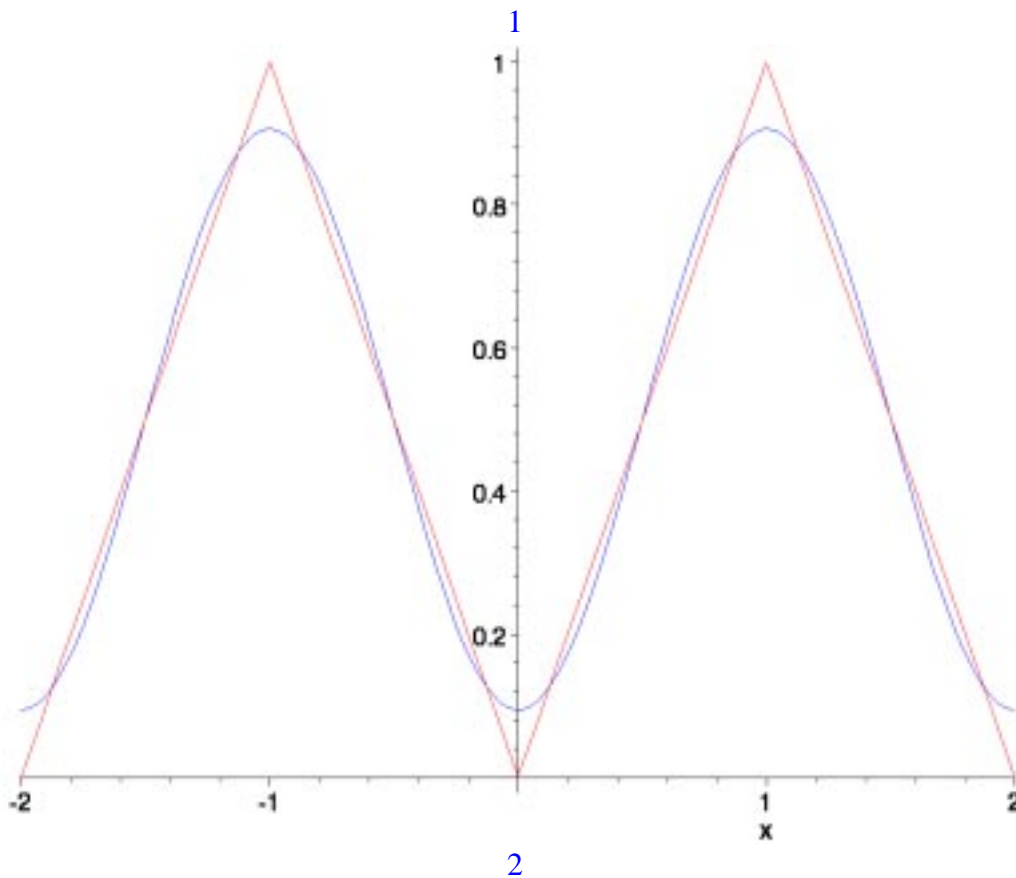
$$a_{15} := -\frac{4}{225} \frac{1}{\pi^2}$$

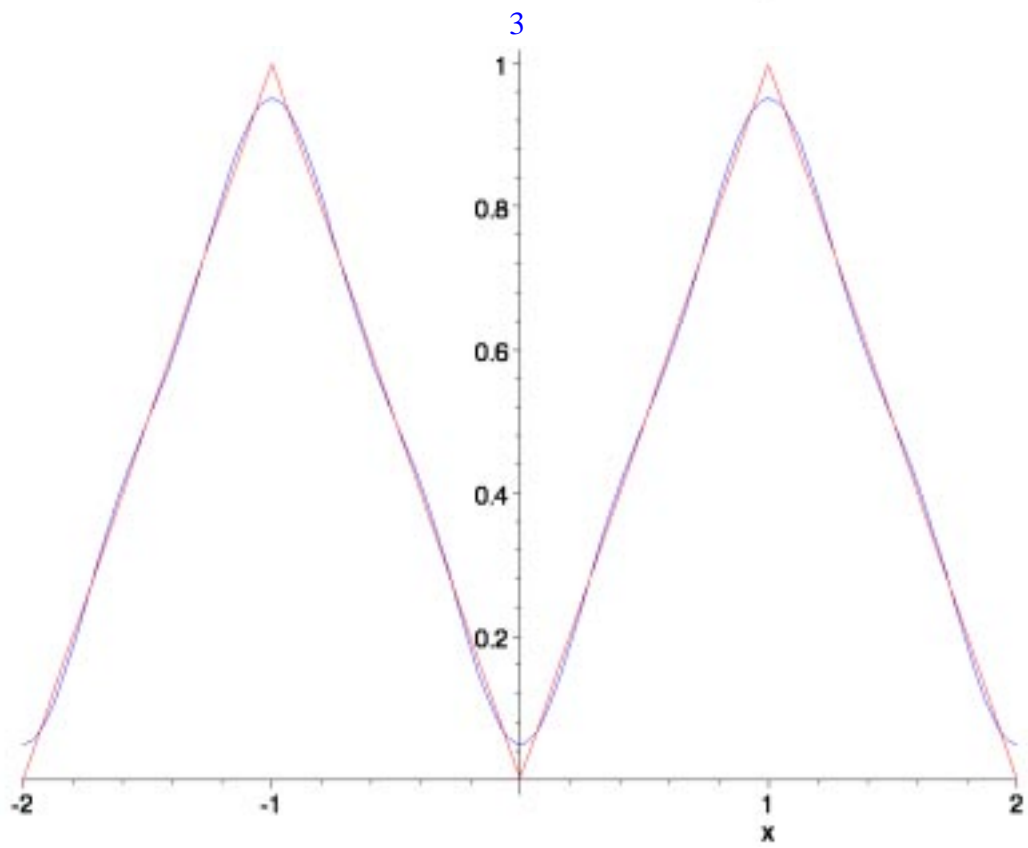
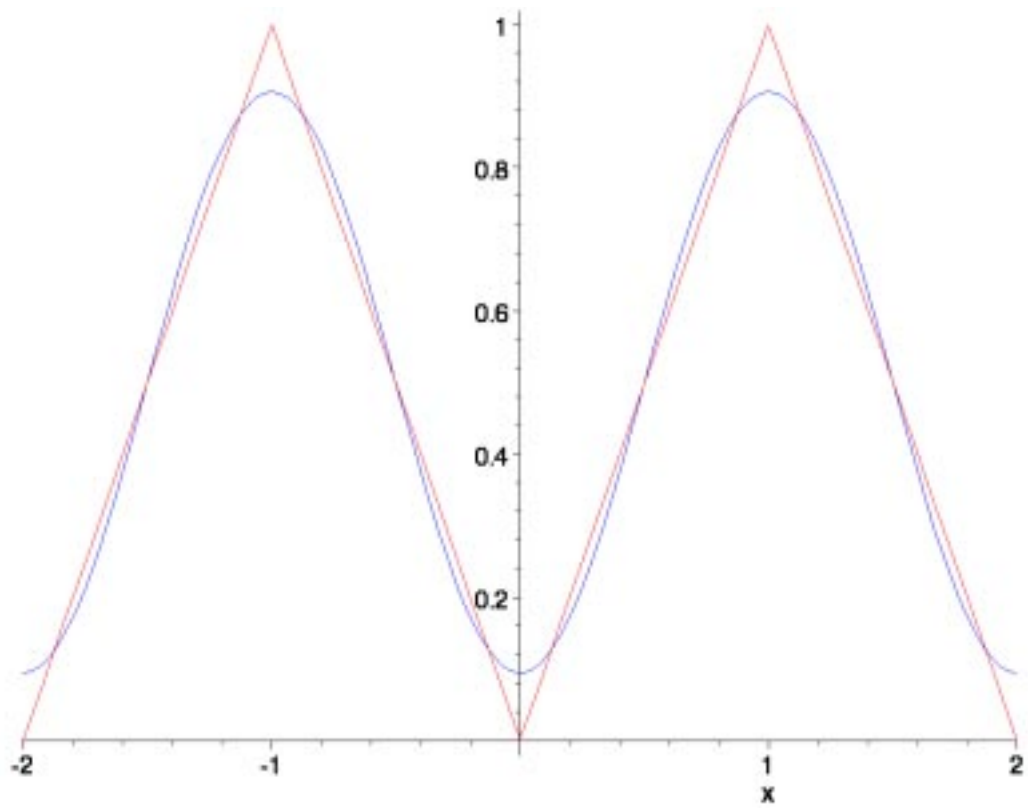
$$b_{15} := 0$$

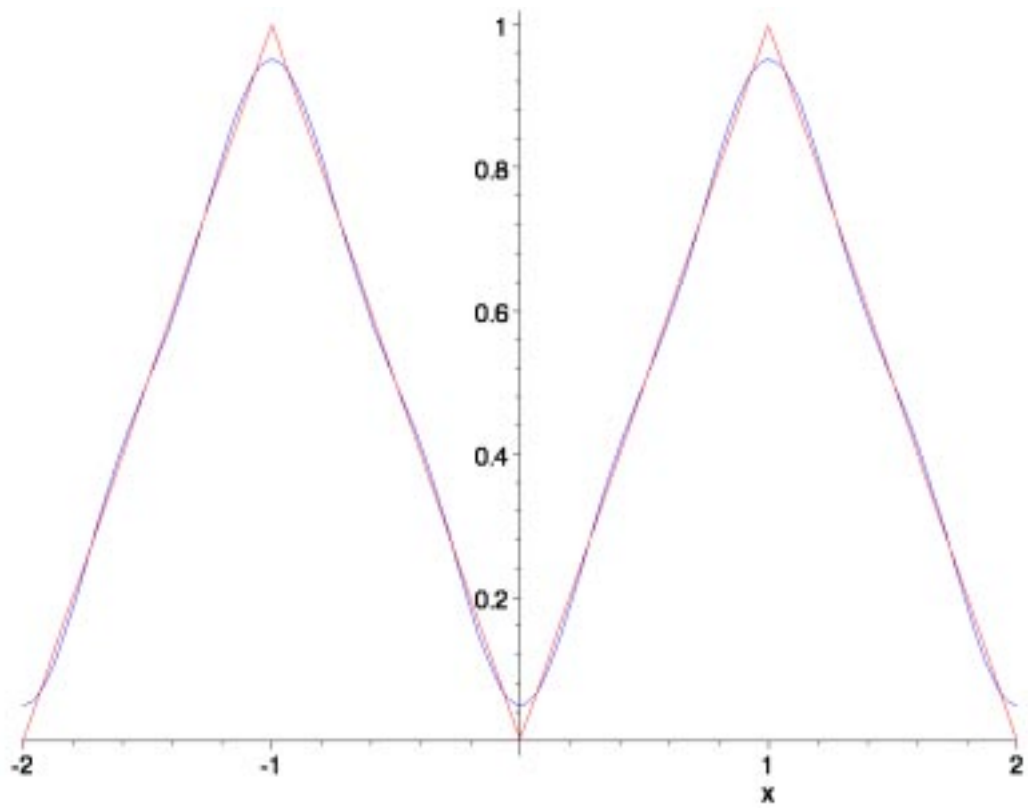
$$F_{15} := \frac{1}{2} - \frac{4 \cos(\pi x)}{\pi^2} - \frac{4 \cos(3 \pi x)}{9 \pi^2} - \frac{4 \cos(5 \pi x)}{25 \pi^2} - \frac{4 \cos(7 \pi x)}{49 \pi^2} - \frac{4 \cos(9 \pi x)}{81 \pi^2} \\ - \frac{4 \cos(11 \pi x)}{121 \pi^2} - \frac{4 \cos(13 \pi x)}{169 \pi^2} - \frac{4 \cos(15 \pi x)}{225 \pi^2}$$

Next we plot $f(x)$ along with each of the $F_n(x)$ for $n = 1..15$.

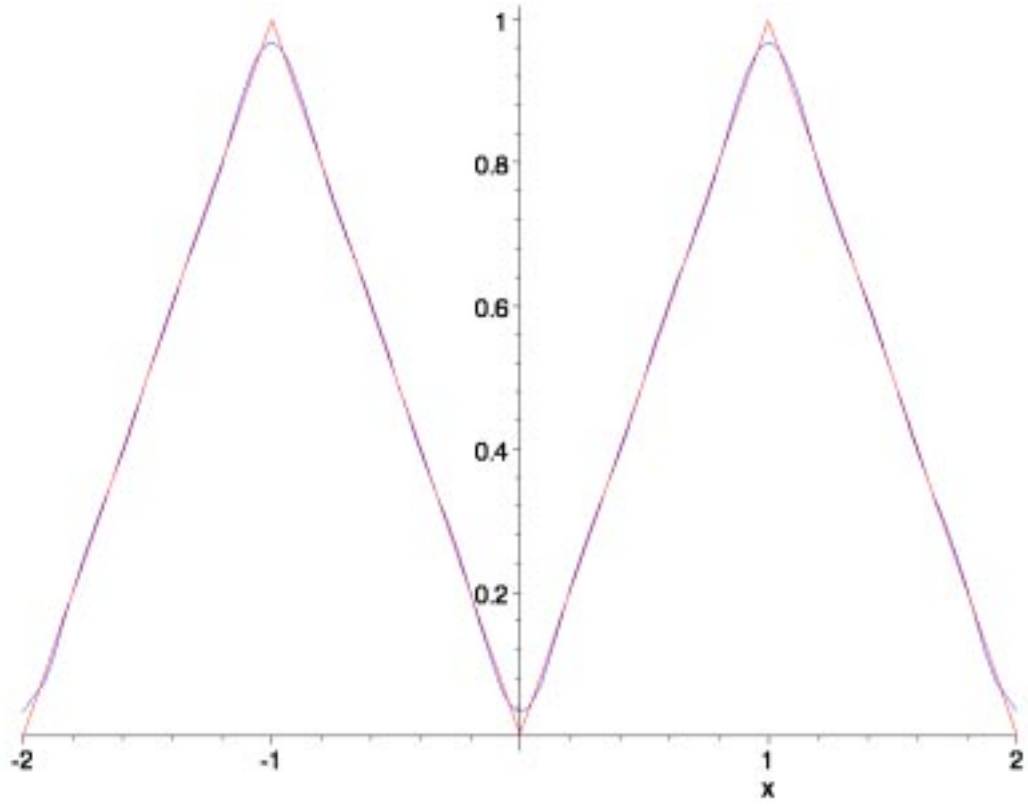
```
> for k from 1 to n do
  p[k]:=plot(F[k],x=-2..2,color=blue):
od:
> for k from 1 to n do
  k;
  display(p[0],p[k]):
od;
```



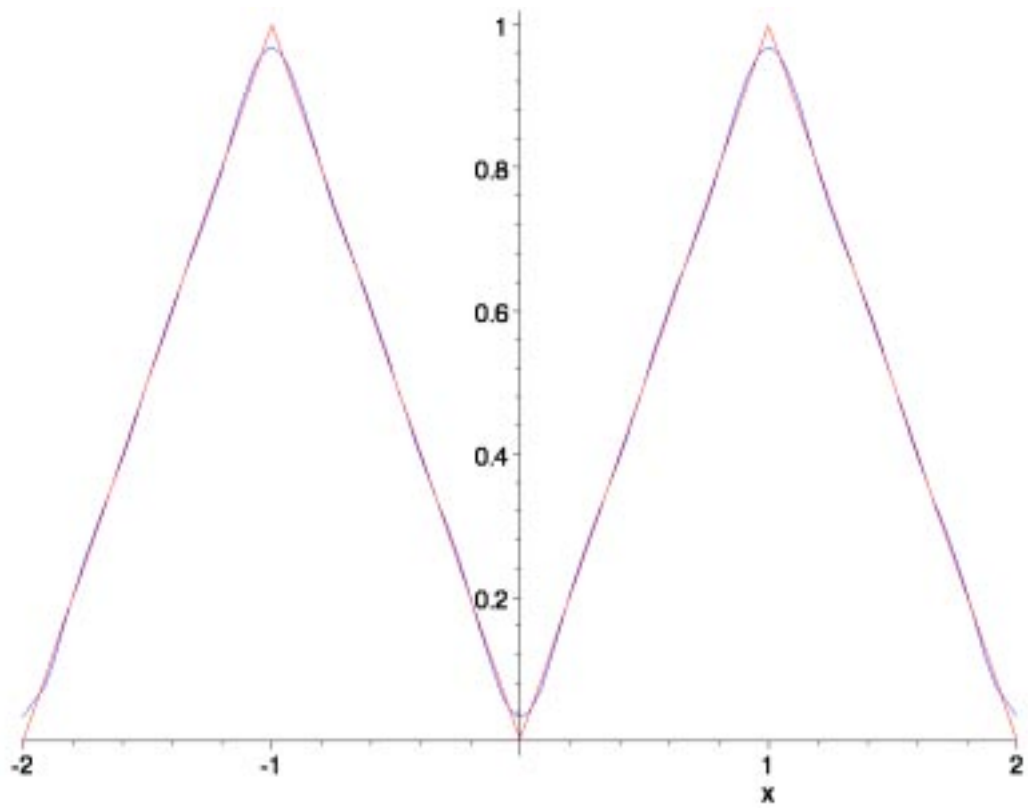




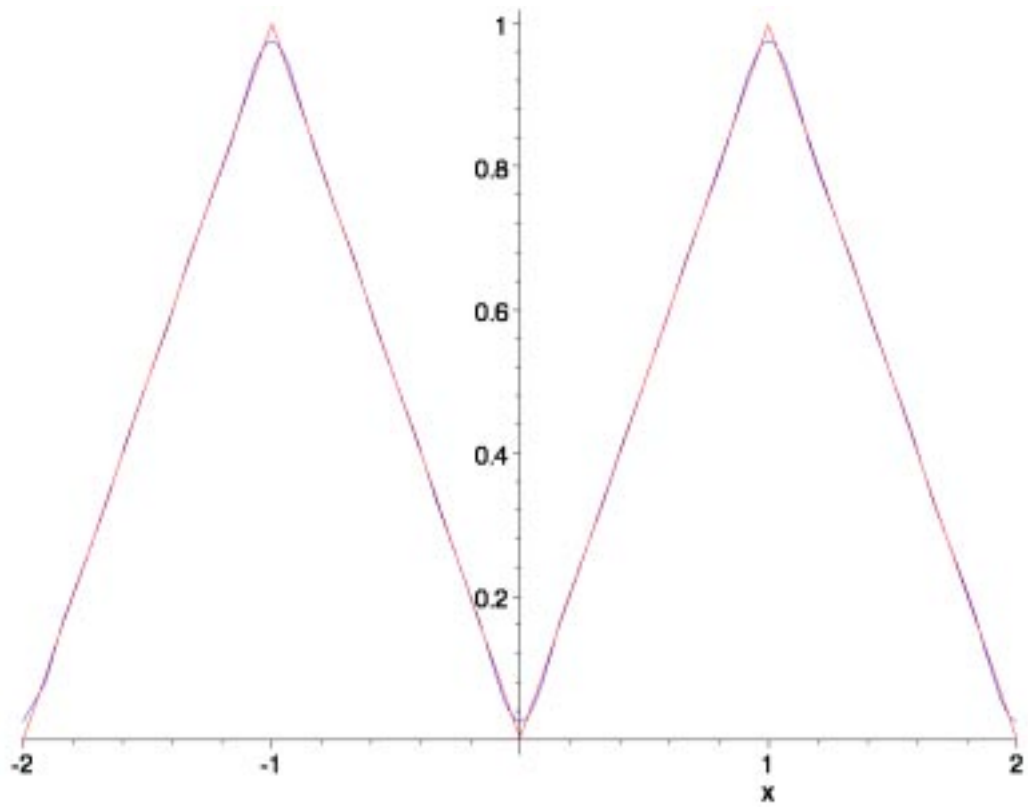
5



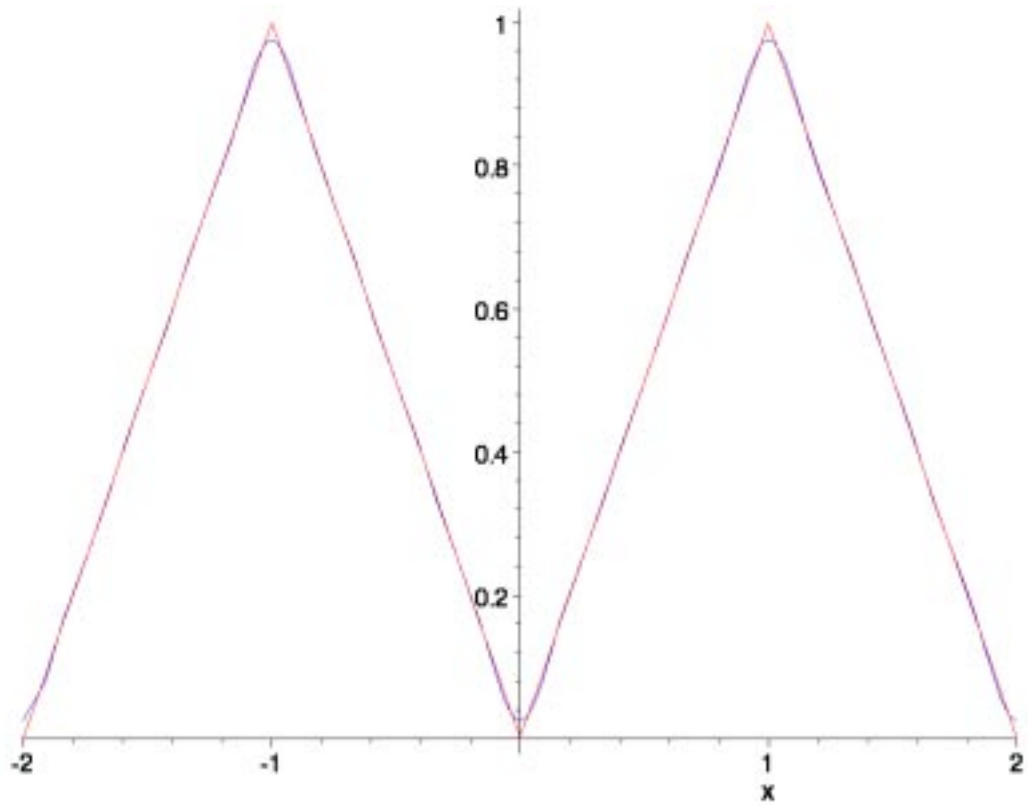
6



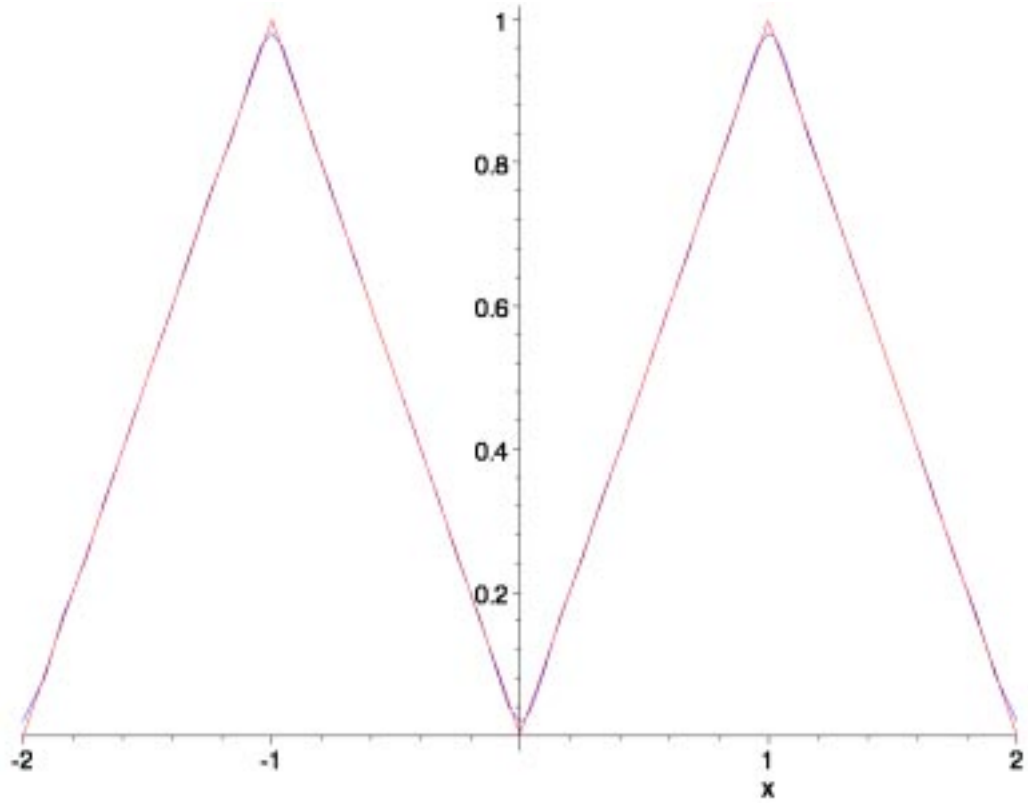
7



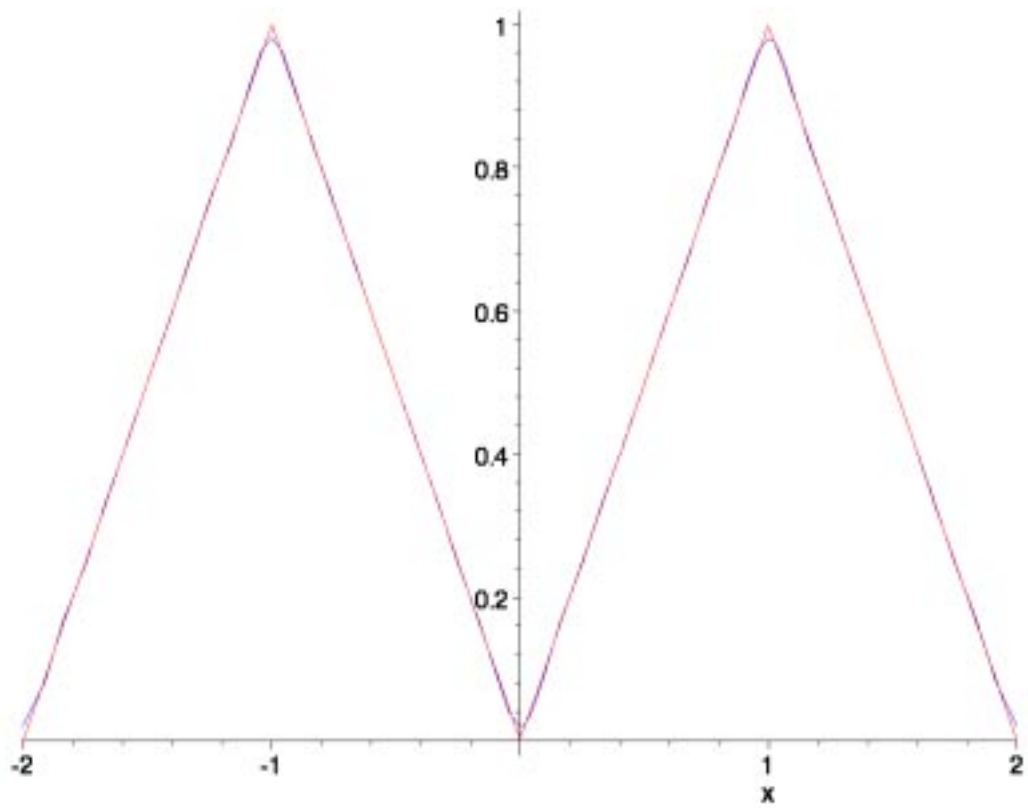
8



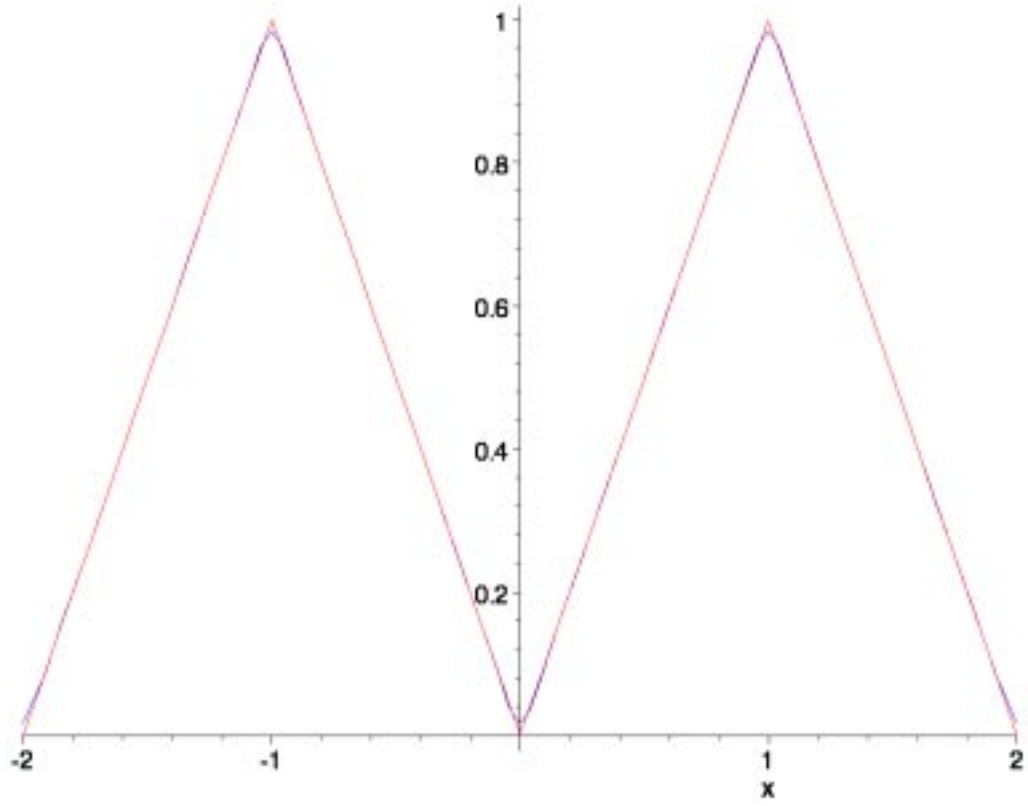
9



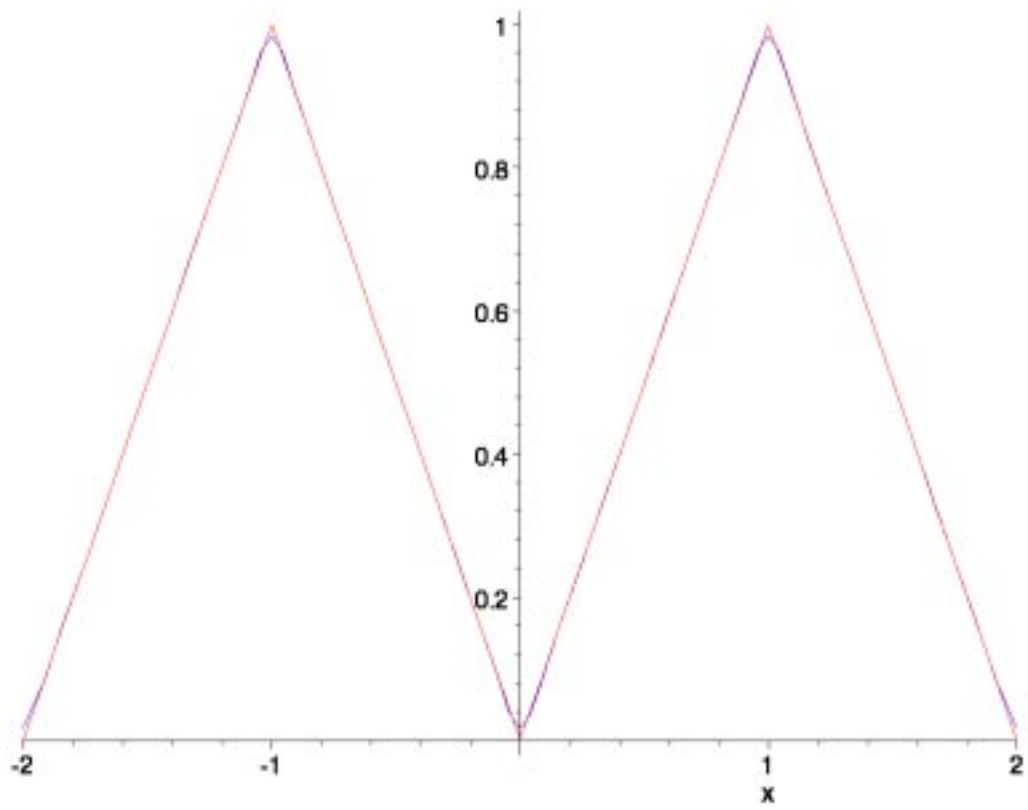
10



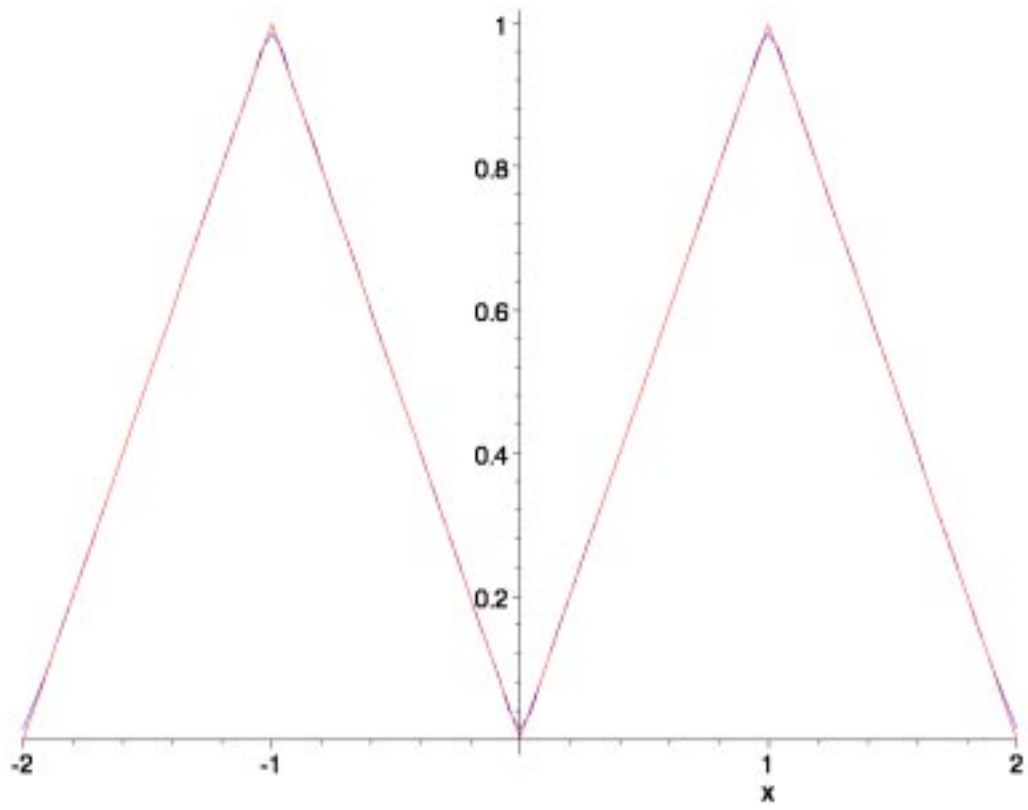
11



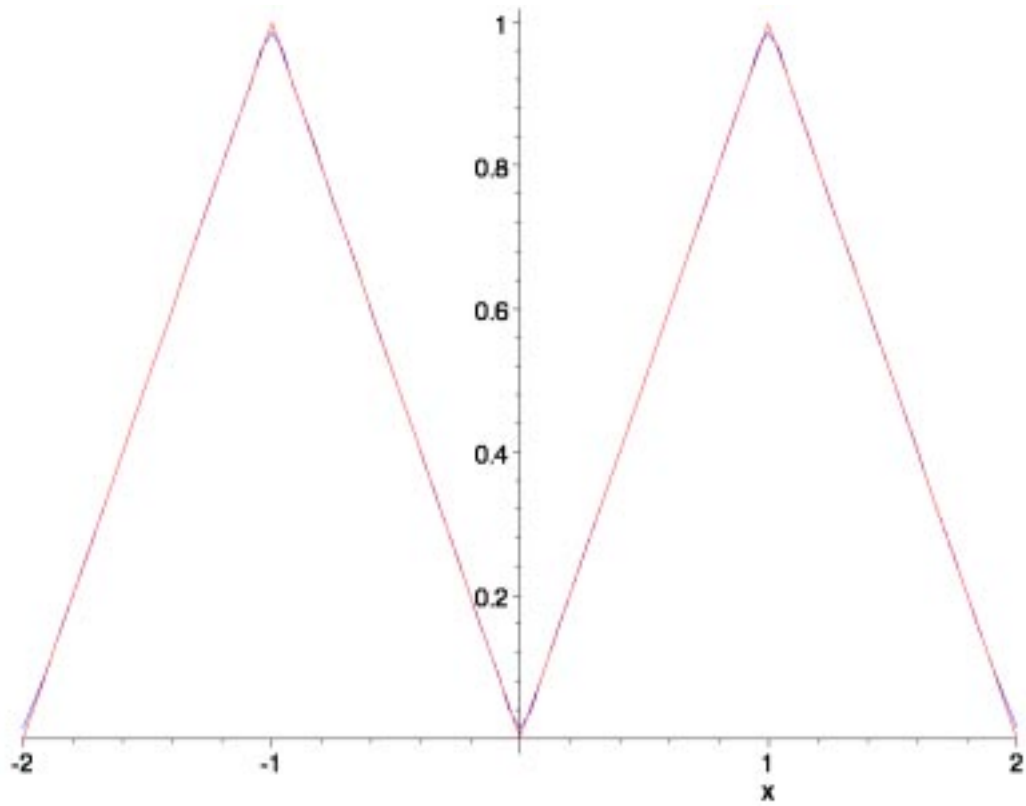
12



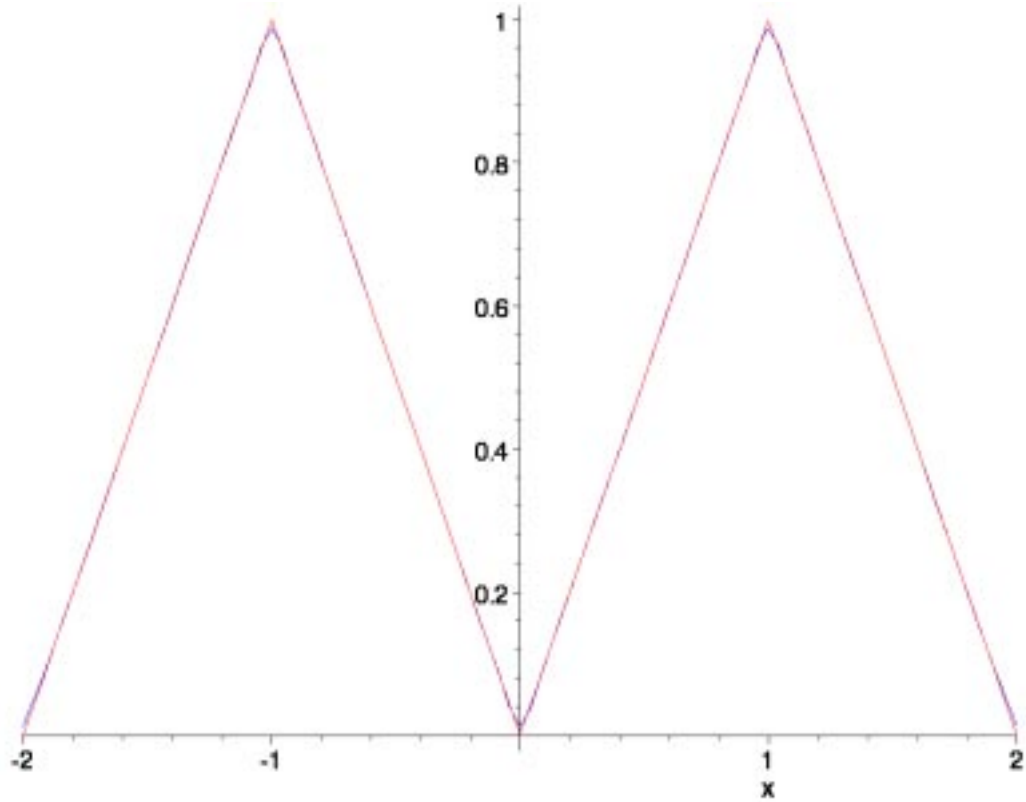
13



14



15



[Again, we see that the successive Fourier polynomials give better and better approximations to $f(x)$.

- [>
- [>