Trigonometric Polynomial Approximations

Continuous

We find the continuous least squares trigonometric polynomial $S_10(x)$ for $f(x)=e^x$ on $[-\pi, \pi]$.

We first enter our value for $n$ and our function.

```maple
> n:=10;

n := 10

> f:=x->exp(x);

f := x -> e^x
```

We compute the Fourier coefficients $a_k$ and $b_k$.

```maple
> for k from 0 to n do
a[k]:=1/Pi*Int(f(x)*cos(k*x),x=-Pi..Pi);
a[k]:=evalf(a[k]);
od;

a_0 := \int_{-\pi}^{\pi} e^x \, dx

a_0 := 7.352155817

a_1 := \int_{-\pi}^{\pi} e^x \cos(x) \, dx

a_1 := -3.67607910

a_2 := \int_{-\pi}^{\pi} e^x \cos(2x) \, dx

a_2 := 1.470431164

a_3 := \int_{-\pi}^{\pi} e^x \cos(3x) \, dx

a_3 := -0.7352155817

a_4 := \int_{-\pi}^{\pi} e^x \cos(4x) \, dx

a_4 := 0.4324797542
```
\[
\int_{-\pi}^{\pi} e^x \cos(5x) \, dx = a_5 := -0.2827752238
\]

\[
\int_{-\pi}^{\pi} e^x \cos(6x) \, dx = a_6 := 0.1987069140
\]

\[
\int_{-\pi}^{\pi} e^x \cos(7x) \, dx = a_7 := -0.1470431164
\]

\[
\int_{-\pi}^{\pi} e^x \cos(8x) \, dx = a_8 := 0.1131100895
\]

\[
\int_{-\pi}^{\pi} e^x \cos(9x) \, dx = a_9 := -0.08966043682
\]

\[
\int_{-\pi}^{\pi} e^x \cos(10x) \, dx = a_{10} := 0.07279362198
\]

> for k from 1 to n-1 do
b[k]:=1/Pi*Int(f(x)*sin(k*x),x=-Pi..Pi);
b[k]:=evalf(b[k]);
od;

\[
\int_{-\pi}^{\pi} e^x \sin(x) \, dx = b_1 := 3.676077910
\]
We now form the degree $n$ trigonometric polynomial over the interval $[-\pi, \pi]$ in the variable $x$. We name
this polynomial $S_n'$. We first reset two variables.

```plaintext
> n:='n'; k:='k';

We first reset two variables.
```

```plaintext
> S[n]:='a[0]/2+a[n]*cos(n*x)+Sum(a[k]*cos(k*x)+b[k]*sin(k*x),k=1..n-1);

We now look specifically at $S_{10}$.

```plaintext
> S[10]:=unapply(a[0]/2+a[10]*cos(10*x)+add(a[k]*cos(k*x)+b[k]*sin(k*x),k=1..10-1),x);

We graph the function (red) and the trigonometric polynomial (green) on the interval $[-\pi, \pi]$.

```plaintext
> plot({f(x),S[10](x)},x=-Pi..Pi);
```
Notice the less than perfect fit. Increasing the degree of the polynomial will improve the fit, but a real good fit may necessitate a very high degree. This should not be surprising since we are fitting a non-period function with a sum of periodic functions.

**Discrete**

We do Example 2 on page 373.

We first enter the value of m such that we have 2m data points \((x_j, y_j)\). We also enter the degree n of our trigonometric polynomial with \(n < m\). The greater we make n the more accurate our polynomial should be.

\[
\begin{align*}
& m := 5; n := 3; \\
& m := 5 \\
& n := 3
\end{align*}
\]

We input the endpoints of our closed interval \([A, B]\).

\[
\begin{align*}
& A := 0; B := 2; \\
& A := 0 \\
& B := 2
\end{align*}
\]

We form the array \(x\) of equally spaced \(x\) values.

\[
\begin{align*}
& \text{for } j \text{ from 0 to } 2m-1 \text{ do} \\
& x[j] := A + j * (B - A) / (2m)
\end{align*}
\]
We enter our y values into the y array either by using a function that generates them or by putting them into a list L and using a for loop to place them into the array.

\[ f := x \rightarrow x^4 - 3x^3 + 2x^2 - \tan(x(x-2)) \]

\[
\begin{align*}
  x_0 &:= 0 \\
  x_1 &:= \frac{1}{5} \\
  x_2 &:= \frac{2}{5} \\
  x_3 &:= \frac{3}{5} \\
  x_4 &:= \frac{4}{5} \\
  x_5 &:= 1 \\
  x_6 &:= \frac{6}{5} \\
  x_7 &:= \frac{7}{5} \\
  x_8 &:= \frac{8}{5} \\
  x_9 &:= \frac{9}{5}
\end{align*}
\]

We use a linear transformation \( z = T(x) = -\pi + (2\pi)(x-A)/(B-A) \) to transform the interval \([A, B]\) into the interval \([-\pi, \pi]\). We use the linear transformation to transform each \( x_j \) in \([A, B]\) into the corresponding
\[ z_j \text{ in } [-\pi, \pi]. \]

\[ \text{eq} := z = -\pi + \pi x. \]

\[ \text{for } j \text{ from 0 to } 2^m - 1 \text{ do} \]
\[ z[j] := -\pi + (2\pi)(x[j] - A)/(B - A) \]
\[ \text{od}; \]

\[ z_0 := -\pi \]
\[ z_1 := -\frac{4}{5}\pi \]
\[ z_2 := -\frac{3}{5}\pi \]
\[ z_3 := -\frac{2}{5}\pi \]
\[ z_4 := -\frac{1}{5}\pi \]
\[ z_5 := 0 \]
\[ z_6 := \frac{1}{5}\pi \]
\[ z_7 := \frac{2}{5}\pi \]
\[ z_8 := \frac{3}{5}\pi \]
\[ z_9 := \frac{4}{5}\pi \]

We next find the coefficients \( a_k \) and \( b_k \).

\[ \text{for } k \text{ from 0 to } n \text{ do} \]
\[ a[k] := \text{evalf}(1/m*\text{add}(y[j]*\cos(k*z[j]), j=0..2^m-1)); \]
\[ \text{od}; \]

\[ a_0 := 1.524016151 \]
\[ a_1 := 0.7717690394 \]
\[ a_2 := 0.0174227900 \]
\[ a_3 := 0.0065672733 \]

\[ \text{for } k \text{ from 1 to } n-1 \text{ do} \]
\[ b[k] := \text{evalf}(1/m*\text{add}(y[j]*\sin(k*z[j]), j=1..2^m-1)); \]
\[ \text{od}; \]

\[ b_1 := -0.3867590453 \]
\[ b_2 := 0.04780604708 \]

We now form the trigonometric polynomial over the interval \([-\pi, \pi]\) in the variable \( z \). We name this polynomial \( s \).

\[ s := z \rightarrow a[0]/2 + a[n]*\cos(n*z) + \text{sum('a[k]*\cos(k*z)+b[k]*\sin(k*z)', 'k' = 1..n-1)}; \]
We translate the trigonometric polynomial back to our original interval \([A, B]\).

\[
Z := \text{rhs(eq)}; \\
S := \text{unapply}(s(Z), x);
\]

\[Z := -\pi + \pi x \]

\[S := x / 0.7620080755 \quad \cos(3x) + 0.3867590453 \sin(x) + 0.0174227900 \cos(2x) + 0.04780604708 \sin(2x)\]

We compute the least squares error.

\[
\text{lserror} := \text{evalf}\left(\text{add}\left(\text{abs}(y[j] - S(x[j]))^2, j = 0..2*m-1\right)\right); \\
\text{lserror} := 0.0009841079475
\]

We graph the function (red) and the trigonometric polynomial (green) on the interval \([A, B]\).

\[
\text{plot}\left\{f(x), S(x)\right\}, x = A..B; \\
\]

\[
\text{Curve 1} \quad \text{Curve 2}
\]