Nested Evaluation of Polynomials

Consider the polynomial function \( f(x) = \frac{4}{15}x^5 + \frac{2}{3}x^4 + \frac{4}{3}x^3 + 2x^2 + 2x + 1 \). Suppose we wish to evaluate it at \( x = -\frac{73}{25} \), i.e., we want to find \( f(-\frac{73}{25}) \). Suppose further that we we will use 3 digit rounding. We define the function.

\[
\text{restart;}
\text{f:=x->(4/15)*x^5+(2/3)*x^4+(4/3)*x^3+2*x^2+2*x+1;}
\]

We find the exact value.

\[
\text{X:=-73/25;}
\]

\[
\text{f(X);}
\]

We round this value to 10 digits.

\[
\text{evalf(\%;)}
\]

Next we evaluate the answer using three digit rounding.

\[
\text{Digits:=3;}
\]

We change our argument to a three digit decimal.

\[
\text{x:=evalf(X);}
\]

We form our powers.

\[
\text{x2:=x*x;x3:=x2*x;x4:=x3*x;x5:=x4*x;}
\]

We multiply the powers by our coefficients.

\[
\text{t5:=(4/15)*x5;t4:=(2/3)*x4;t3:=(4/3)*x3;t2:=2*x2;t1:=2*x;}
\]
Now we add the terms.

\[ p := t_5 + t_4; p := p + t_3; p := p + t_2; p := p + t_1; p := p + 1; \]

\[
p := -8.0
\]
\[
p := -41.2
\]
\[
p := -24.1
\]
\[
p := -29.9
\]
\[
p := -28.9
\]

Notice that we needed to do 9 multiplications and 5 additions. Now we set Digits to 10 and find the relative error.

\[
> \text{Digits} := 10;
\]
\[
\text{Digits} := 10
\]
\[
> \text{rele} := \text{abs}(f(X) - p) / \text{abs}(f(X));
\]
\[
\text{rele} := 0.007747840987
\]

Thus our approximation has 2 significant digits.

We can also use the \texttt{RelativeError} command from the \texttt{NumericalAnalysis} package to find the relative error.

\[
> \text{with(Student[NumericalAnalysis])};
\]
\[
[\text{AbsoluteError}, \text{AdamsBashforth}, \text{AdamsBashforthMoulton}, \text{AdamsMoulton}, \text{AdaptiveQuadrature},
\text{AddPoint}, \text{ApproximateExactUpperBound}, \text{ApproximateValue}, \text{BackSubstitution}, \text{BasisFunctions},
\text{Bisection}, \text{CubicSpline}, \text{DataPoints}, \text{Distance}, \text{DividedDifferenceTable}, \text{Draw}, \text{Euler}, \text{EulerTutor},
\text{ExactValue}, \text{FalsePosition}, \text{FixedPointIteration}, \text{ForwardSubstitution}, \text{Function},
\text{InitialValueProblem}, \text{InitialValueProblemTutor}, \text{Interpolant}, \text{InterpolantRemainderTerm},
\text{IsConvergent}, \text{IsMatrixShape}, \text{IterativeApproximate}, \text{IterativeFormula}, \text{IterativeFormulaTutor},
\text{LeadingPrincipalSubmatrix}, \text{LinearSolve}, \text{LinearSystem}, \text{MatrixConvergence},
\text{MatrixDecomposition}, \text{MatrixDecompositionTutor}, \text{ModifiedNewton}, \text{NevilleTable}, \text{Newton},
\text{NumberOfSignificantDigits}, \text{PolynomialInterpolation}, \text{Quadrature}, \text{RateOfConvergence},
\text{RelativeError}, \text{RemainderTerm}, \text{Roots}, \text{RungeKutta}, \text{Secant}, \text{SpectralRadius}, \text{Steffensen}, \text{Taylor},
\text{TaylorPolynomial}, \text{UpperBoundOfRemainderTerm}, \text{VectorLimit}]\]
\[
> \text{RelativeError}(p, f(X));
\]
\[
0.007808338754
\]

Differences from the above can be attributed to round-off schemes.

Next we write our polynomial in nested form, i.e., we write it as

\[
f(x) = \left( \left( \left( \frac{4x}{15} + \frac{2}{3} \right)x + \frac{4}{3} \right)x + 2 \right)x + 1
\]

We again evaluate using three digit rounding.

\[
> \text{Digits} := 3;
\]
\[
\text{Digits} := 3
\]
\[
> p := (4/15)*x; p := p + 2/3; p := p*x; p := p + 4/3; p := p*x; p := p + 2; p := p*x; p := p + 1;
\]
\[
p := -0.779
\]
Notice that we now only needed 5 multiplications along with 5 additions. Let's check the relative error with this approach.

```
Digits:=10;
```

```
Digits := 10
```

```
> RelativeError(p,f(X));
```

```
0.0008818209622
```

Not only is the nested approach more efficient, but our approximation now has 3 significant digits.