Chapter 10

The Laplace Transform

A few worked examples should convince the reader that the Laplace transform furnishes a useful technique for solving linear differential equations. (However, the study of differential equations is no more contained in a table of Laplace transforms than the study of geometry in a set of trigonometric tables.)

T. W. Körner

Sire, I had no need of that hypothesis.

Pierre-Simon de Laplace

10.1 Introduction to the Laplace Transform

Many, if not most, engineers in academia and industry prefer solving linear differential equations and systems with a method involving a certain type of improper integral known as the Laplace transform. Although we will use this method for only first- and second-order linear equations, it can also be used to solve higher-order linear equations. The goal of this section is to introduce the notion of the Laplace transform and to formally define it.

We will introduce the Laplace transform as an alternative way of solving first-order linear equations. This will establish right away the effectiveness of the method.

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1See [35, p. 378]. Thomas William Körner is a Cambridge University mathematician who has written research papers and books on Fourier analysis, including a popular textbook for engineers, physicists, and mathematicians entitled An Introduction to Fourier Analysis. This particular quote comes from another of his books, Fourier Analysis (Cambridge University Press), which consists of essays on the ideas and results of Fourier analysis and their applications to other disciplines.

2See [10, ch. 11]. Pierre-Simon de Laplace (1749–1827) was a French mathematician and astronomer. All of his mathematical research in celestial mechanics, such as applying the laws of Newton to model the motions of the planets, were bound together under the title of the Mécanique Céleste. He also held political offices and was made a count by Napoleon Bonaparte. The above quote was in response to Napoleon’s remark: “You have written this huge book on the system of the world without once mentioning the author of the universe.” More on Laplace and other famous mathematicians can be found in the classic book Men of Mathematics by E. T. Bell [10], from which the above quote was taken.
However, instead of introducing the Laplace transform in this way, we could begin with the formal definition of the Laplace transform right away, after which we could state and prove a number of its properties. Then we could determine the Laplace transforms of the most commonly occurring functions in engineering. But even after all of this, you probably would still fail to see any connection between the Laplace transform and differential equations. Eventually we would get around to showing how to use the Laplace transform to solve differential equations. This is the approach of many textbooks. However efficient this may be in getting students to solve equations, it may still leave them wondering how anyone ever came up with the method in the first place. Actually the development and use of the Laplace transform was a lengthy process. It was not the work of any one person and did not occur over the course of a few days, as a reader might construe from studying a textbook on the subject. The Laplace transform, an improper integral, was first introduced by Pierre-Simon de Laplace in *Théorie Analytique des Probabilités*, a treatise on probability published in 1812. However, Laplace did not have the last word on the subject. Its application to differential equations was still being developed as late as the early 1930s by G. Doetsch.

Our approach to the subject will be to ask whether there is another way to solve first-order linear differential equations besides the methods seen in earlier chapters. We begin our investigation with the initial value problem

$$\frac{dx}{dt} + 2x = 3e^t, \quad x(0) = 1.$$  \hspace{1cm} (10.1.1)

Since the differential equation is already in standard form, its solutions are readily found by multiplying it by the integrating factor $e^{2t}$ and then integrating the result:

$$e^{2t} \frac{dx}{dt} + 2e^{2t}x = 3e^{3t} \quad \Rightarrow \quad \frac{d}{dt} \left( e^{2t}x \right) = 3e^{3t}$$

$$\Rightarrow \quad \int_0^t \frac{d}{dt} \left( e^{2t}x \right) \, dt = \int_0^t 3e^{3t} \, dt + C \quad (10.1.2)$$

Solving for $x$, we obtain the general solution of the differential equation:

$$x(t) = e^t + Ce^{-2t}$$

Setting $t = 0$ and using the initial condition $x(0) = 1$ allows us to determine the value of $C$:

$$x(0) = 1 + C = 1 \quad \Rightarrow \quad C = 0.$$ 

Therefore, the solution of the initial value problem is

$$x(t) = e^t. \quad (10.1.3)$$

However, as we have seen before, there is usually more than one way to attack a problem. Suppose we take a different approach by ignoring that the left-hand side of the first equation in (10.1.2) is the derivative of a product. Instead let’s integrate each of the terms of (10.1.2) from $t = 0$ to $t = T$ to see if somehow

$$\int_0^T e^{2t} x'(t) \, dt + \int_0^T 2e^{2t} x(t) \, dt = \int_0^T 3e^{3t} \, dt \quad (10.1.4)$$
10.1. INTRODUCTION TO THE LAPLACE TRANSFORM

will yield the solution of (10.1.1). Let’s integrate the first integral on the left-hand side by parts. First set \( u = e^{2t} \) and \( dv = x'(t) \, dt \); then \( du = 2e^{2t} \, dt \) and \( v = x(t) \). Thus,

\[
\int e^{2t} x'(t) \, dt = \int u \, dv = uv - \int v \, du = e^{2t} x(t) - \int 2e^{2t} x(t) \, dt,
\]

and so

\[
\int_0^T e^{2t} x'(t) \, dt = \left[ e^{2t} x(t) \right]_0^T - \int_0^T 2e^{2t} x(t) \, dt.
\]

Thus, (10.1.4) becomes

\[
e^{2T} x(T) - x(0) - 2\int_0^T e^{2t} x(t) \, dt + 2\int_0^T e^{2t} x(t) \, dt = e^{3T} - 1. \tag{10.1.5}
\]

Since the initial condition is \( x(0) = 1 \), this simplifies to \( x(T) = e^T \). We obtain (10.1.3), the solution \( x(t) = e^t \), by merely replacing \( T \) with \( t \).

Now let’s look at a variation of the previous approach that is more general in that we multiply both sides of the differential equation by the exponential function \( e^{-st} \) but do not assign a value to \( s \). If \( e^{-st} \) is an integrating factor when \( s = -2 \), we ignore this fact to see if we can still obtain the solution of (10.1.1). As before, we integrate each term of the resulting equation with respect to \( t \) from \( t = 0 \) to \( t = T \):

\[
\int_0^T e^{-st} x'(t) \, dt + \int_0^T 2e^{-st} x(t) \, dt = \int_0^T 3e^{-st} e^t \, dt. \tag{10.1.6}
\]

Note that this reduces to (10.1.4) when \( s = -2 \). Similar to our handling of (10.1.4), we integrate the first integral on the left-hand side of (10.1.6) by parts by setting \( u = e^{-st} \) and \( dv = x'(t) \, dt \) obtaining

\[
\left[ e^{-st} x(t) \right]_{t=0}^{t=T} - \int_0^T (-s) e^{-st} x(t) \, dt + \int_0^T 2e^{-st} x(t) \, dt = \int_0^T 3e^{-st} e^t \, dt
\]

which simplifies to

\[
e^{-sT} x(T) - x(0) + (s + 2) \int_0^T e^{-st} x(t) \, dt = -\frac{3}{s-1} \left[ e^{-(s-1)T} - 1 \right]. \tag{10.1.7}
\]

Note that this reduces to (10.1.5) when \( s = -2 \), but again we ignore the efficacy of assigning \( s \) this value. Rather we take a different approach and that is to let \( T \) increase without bound; symbolically, to let \( T \to \infty \). If \( s < 1 \), the right-hand side of (10.1.7) diverges; but if \( s \) is restricted to values greater than 1, it converges to

\[
\frac{3}{s-1}.
\]

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3The reason for using \( e^{-st} \) instead of \( e^{st} \) is dictated by tradition. We will be concerned shortly with the convergence of improper integrals of the form \( \int_0^\infty e^{-st} x(t) \, dt \). In this particular example, as it turns out, these integrals converge for \( s > 1 \). If tradition had instead chosen \( e^{st} \), then that would have merely changed the form of the improper integrals to \( \int_0^\infty e^{st} x(t) \, dt \); and in this example, convergence would be for \( s < 1 \).
since $e^{-(s-1)T} \to 0$ as $T \to \infty$. In short, when we take the limit of both sides of (10.1.6) as $T \to \infty$ restricting $s$ to $s > 1$, the right-hand side of (10.1.6) becomes

$$\lim_{T \to \infty} \int_0^T 3e^{-st} \, dt = 3 \lim_{T \to \infty} \int_0^T e^{-(s-1)t} \, dt$$

$$= -\frac{3}{s-1} \lim_{T \to \infty} \left[ e^{-(s-1)T} - 1 \right] = \frac{3}{s-1}. \quad (10.1.8)$$

This is what is meant by writing

$$\int_0^\infty 3e^{-st} \, dt = \frac{3}{s-1}.$$ 

The integral $\int_0^\infty 3e^{-st} \, dt$ is called an improper integral because $\infty$ is not a real number in the usual sense; as a result, it must be treated with a little more care than the usual definite integrals both of whose limits of integration are real numbers. Recall from calculus that

$$\int_a^\infty f(t) \, dt$$

is really mathematical shorthand for

$$\lim_{T \to \infty} \int_a^T f(t) \, dt.$$ 

Let’s also assume that $e^{-sT} x(T) \to 0$ as $T \to \infty$. This will be justified shortly. So, as $T \to \infty$, (10.1.7) becomes

$$-x(0) + (s + 2) \int_0^\infty e^{-st} x(t) \, dt = \frac{3}{s-1}. \quad (10.1.9)$$

Setting $x(0) = 1$ and solving for the improper integral yields

$$\int_0^\infty e^{-st} x(t) \, dt = \frac{3}{(s-1)(s+2)} + \frac{1}{s+2},$$

which simplifies to

$$\int_0^\infty e^{-st} x(t) \, dt = \frac{1}{s-1}. \quad (10.1.10)$$

At first this result may seem interesting yet useless: we have not explicitly solved for $x(t)$; instead we have what is known as an integral transform of $x(t)$. But then our hopes are rekindled when we recall that the quantity $1/(s-1)$ also appeared in a previous integration, namely (10.1.8):

$$\int_0^\infty e^{-st} \, dt = \frac{1}{s-1}. $$
10.1. INTRODUCTION TO THE LAPLACE TRANSFORM

Even though we may not know how to find solutions of (10.1.10) directly, it does not really matter, for we do know from (10.1.8) that the differentiable function \( x(t) = e^t \) is a solution. Furthermore, the assumption that led to this result, namely that

\[
\lim_{T \to \infty} e^{-sT} x(T) = 0 \quad \text{as} \quad T \to \infty,
\]

turns out to be valid since

\[
e^{-sT} x(T) = e^{-sT} e^T = e^{-(s-1)T}
\]
does approach 0 as \( T \to \infty \) for \( s > 1 \). And so we have managed to obtain the solution of (10.1.1) for a third time!

Now the author must confess that this initial value problem was contrived so that the solution of (10.1.10) just happened to be around, namely, in (10.1.8). Normally we won’t be so lucky. For example, if the initial condition is changed from \( x(0) = 1 \) to \( x(0) = 2 \), then (10.1.8) remains as it is; however, (10.1.10) becomes

\[
\int_0^\infty e^{-st} x(t) \, dt = \frac{3}{(s-1)(s+2)} + \frac{2}{s+2} = \frac{2s+1}{(s-1)(s+2)}.
\]

If this approach to solving linear differential equations is to be effective and an alternative to the integrating factor method, we need some way of ascertaining functions \( x(t) \) that will satisfy

\[
\int_0^\infty e^{-st} x(t) \, dt = X(s)
\]

(10.1.11)

when \( X(s) \) is a known function of \( s \). The function \( X(s) \) is called an integral transform of \( x(t) \). Actually, there are many well-known integral transforms—this particular one is called the Laplace transform. And right now we have our first example: the Laplace transform of \( x(t) = e^t \) for \( t \geq 0 \) is

\[
X(s) = \frac{1}{s-1}
\]

for \( s > 1 \).

Before continuing, let us formally define what is meant by the Laplace transform.

**The Laplace Transform**

**Definition 10.1.** Let \( f(t) \) be a function on \([0, \infty)\). The Laplace transform of \( f \) is the function \( F \) whose value at a real number \( s \) is defined by

\[
F(s) := \int_0^\infty e^{-st} f(t) \, dt,
\]

(10.1.12)

provided the improper integral converges. The domain of \( F \) consists of all the values of \( s \) where the integral converges.

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4The advantages over the integrating factor method will be discussed later after we have fully discussed this new method of solving initial value problems.

5Another well-known integral transform that you may have already encountered is the Fourier transform.
A word about notation: note the use of the lowercase $f$ and the uppercase $F$ in (10.1.12). When Laplace transforms are involved, the convention is to use a lowercase letter for the original function and the uppercase form of the same letter to denote its Laplace transform. For instance, just as we used $X$ to denote the Laplace transform of the function $x(t) = e^t$, we would use $F$ and $G$ to denote the Laplace transforms of functions called $f$ and $g$, respectively.

There is even another way of denoting the Laplace transform of a function $f(t)$ and that is to write $L\{f(t)\}$. For example, either

$$L\{e^t\}(s) = \frac{1}{s-1} \quad \text{or} \quad L\{x(t)\}(s) = \frac{1}{s-1}$$

expresses that the Laplace transform of $x(t) = e^t$ is $X(s) = 1/(s-1)$. It is even acceptable to shorten this to

$$L\{e^t\} = \frac{1}{s-1}.$$  

This might be considered an abuse of notation: for just as we write $X = \frac{1}{s-1}$ and not $X = \frac{1}{s-1}$,

we should write

$$L\{e^t\}(s) = \frac{1}{s-1} \quad \text{instead of} \quad L\{e^t\} = \frac{1}{s-1}.$$  

However, since writing the argument “$s$” in all of the steps of a problem is tedious and as it is already understood that $L\{f(t)\}$ is a function of $s$, we usually omit it.

When taking the Laplace transform of a function, the original function $f(t)$ is transformed into a new function called $L\{f(t)\}$ or $F(s)$. We portray this symbolically by writing

$$f(t) \xrightarrow{L} F(s).$$

Looking at it like this, we can say that the Laplace transform is a rule $L$ that assigns to the original function some other function—but only one other function. Recall that we have seen rules like this before: they are called operators.

The operator $L$ not only transforms a function $f(t)$ into another function $F(s)$, it is also accompanied by a change in domains. For example, we just determined that the Laplace transform of $x(t) = e^t$ for $t \geq 0$ is the function

$$L\{x(t)\} = \frac{1}{s-1} \quad \text{for} \quad s > 1.$$  

Associated with this transform is a change from the $t$-domain $[0, \infty)$ for $f(t)$ to the $s$-domain $(1, \infty)$ for $F(s)$. In applications involving time, where $t$ denotes the time, the $t$-domain is called the time domain.

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6A caveat: Not everyone follows this convention. In some books, it is reversed: the uppercase letter denotes the original function and the corresponding lowercase letter denotes its transform.
The task of determining pairs of functions satisfying (10.1.12) that will be useful to us is not as daunting as it might first seem. The reason for this is that the aim of our investigation is not to find functions \( f(t) \) satisfying (10.1.12) for every imaginable function \( F(s) \); rather only for those \( F(s) \) that correspond to functions \( f(t) \) which are actual solutions of linear differential equations. From our prior experiences with linear equations, these are the exponential, sinusoidal, and algebraic functions, along with certain combinations of these functions. So the game plan will be to substitute these types of functions into (10.1.12) for \( f(t) \) and then work out the resulting improper integral to determine its Laplace transform \( F(s) \). In this way, we can first create a list of useful Laplace transforms and then see where to go from there.

## 10.2 Finding Laplace Transforms with the Definition

In the first section, we found that \( L\{e^t\} = 1/(s - 1) \), which is the succinct way of saying that the Laplace transform of the function \( f(t) = e^t \) is the function

\[
F(s) = \frac{1}{s - 1}.
\]

For our first example in using Definition 10.1, let’s generalize this result by finding the Laplace transform of the function \( f(t) = e^{bt} \), where \( b \) represents any real number.

**Example 10.1.** Find \( L\{e^{bt}\} \), where \( b \) is a constant.

**Solution.** By (10.1.12), the definition of the Laplace transform,

\[
L\{e^{bt}\} = \int_0^\infty e^{-st}e^{bt} \, dt = \lim_{T \to \infty} \int_0^T e^{(b-s)t} \, dt
\]

\[
= \lim_{T \to \infty} \frac{1}{b-s} e^{(b-s)t} \bigg|_{t=0}^{t=T} = \frac{1}{b-s} \lim_{T \to \infty} \left[ e^{(b-s)T} - e^0 \right].
\]

Implicit in carrying out the integration is that \( s \neq b \). As \( T \to \infty \),

\[
e^{(b-s)T} \to 0 \quad \text{if} \quad s > b
\]

whereas

\[
e^{(b-s)T} \to \infty \quad \text{if} \quad s < b.
\]

From this, we conclude that the integral converges to \( 1/(s - b) \) if \( s > b \) and diverges if \( s < b \). For the case \( s = b \),

\[
L\{e^{bt}\} = \int_0^\infty e^{-bt}e^{bt} \, dt = \lim_{T \to \infty} \int_0^T 1 \, dt = \lim_{T \to \infty} T = \infty.
\]

Therefore,

\[
L\{e^{bt}\} = \begin{cases} 
\frac{1}{s-b}, & \text{for } s > b \\
\text{undefined}, & \text{for } s \leq b.
\end{cases} \quad (10.2.1)
\]
This is the same as saying that the Laplace transform of \( f(t) = e^{bt} \) is the function \( F(s) = 1/(s-b) \) with domain \((b, \infty)\).

**Example 10.2.** Show that
\[
\mathcal{L}\{1\} = \frac{1}{s}
\]
for \( s > 0 \).

**Solution.** The “1” really stands for a function whose value at every \( t \geq 0 \) is 1. In other words, this is a succinct way of saying that we are to find the Laplace transform of a function \( f \), where \( f(t) = 1 \) for \( t \geq 0 \). By (10.1.12),
\[
\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt \quad \text{for } s > 0.
\]

If \( s = 0 \), the integral diverges. Therefore, for \( s > 0 \),
\[
\mathcal{L}\{1\} = \frac{1}{s}.
\]

Note that Example 10.1 also implies this. Simply set \( b = 0 \) in (10.2.1):
\[
\mathcal{L}\{1\} = \left. \frac{1}{s-b} \right|_{b=0} = \frac{1}{s}
\]
for \( s > 0 \).

**Example 10.3.** Find the Laplace transform of the unit step function \( u(t - b) \), where \( b \geq 0 \) and
\[
u(t - b) = \begin{cases} 0, & 0 \leq t < b \\ 1, & t > b \end{cases}
\]

**Solution.** By (10.1.12), \( \mathcal{L}\{u(t - b)\} = \int_0^\infty e^{-st} u(t - b) \, dt \). Since the value of the function changes at \( t = b \), we divide the improper integral into two integrals:
\[
\mathcal{L}\{u(t - b)\} = \int_0^b e^{-st} \cdot 0 \, dt + \int_b^\infty e^{-st} \cdot 1 \, dt = \lim_{T \to \infty} \int_0^T e^{-st} \, dt
\]

for \( s > 0 \).
10.2. FINDING LAPLACE TRANSFORMS WITH THE DEFINITION

We conclude
\[ \mathcal{L}\{u(t - b)\} = \frac{e^{-bs}}{s} \quad \text{for } s > 0. \]

Finally note that for \( b = 0 \), \( \mathcal{L}\{u(t)\} = 1/s \), which agrees with Example 10.2.

The next two examples demonstrate the indispensability of the method of integration by parts as a tool in determining Laplace transforms. The first one also demonstrates the importance of recognizing patterns.

**Example 10.4.** For \( n = 1, 2, 3, \ldots \), show that
\[ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}. \]

**Solution.** By Definition 10.1,
\[ \mathcal{L}\{t^n\} = \lim_{T \to \infty} \int_0^T e^{-st} t^n \, dt. \]

This is one of those classic integration-by-parts integrand. By the \textit{LIATE} mnemonic, we set \( u = t^n \) and \( dv = e^{-st} \, dt \). Then \( du = n t^{n-1} \, dt \) and \( v = -\frac{1}{s} e^{-st} \). Thus,
\[ \int_0^T e^{-st} t^n \, dt = -\frac{1}{s} e^{-st} t^n \bigg|_0^T - \int_0^T \left(-\frac{1}{s} e^{-st}\right) n t^{n-1} \, dt \]
\[ = -\frac{1}{s} e^{-sT} T^n + \frac{n}{s} \int_0^T e^{-st} t^{n-1} \, dt. \]

First let us determine the long-term behavior of the first term by using the Maclaurin series expansion of \( e^x \):
\[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots. \]

Recall that this series converges to \( e^x \) regardless of the value of \( x \). Setting \( x = st \) we have
\[ \frac{T^n}{e^{sT}} = \frac{T^n}{1 + sT + \frac{s^2 T^2}{2!} + \frac{s^3 T^3}{3!} + \ldots + \frac{s^n T^n}{n!} + \frac{s^{n+1} T^{n+1}}{(n + 1)!} + \frac{s^{n+2} T^{n+2}}{(n + 2)!} + \ldots} \]
\[ = \frac{1}{\frac{T^n}{T^n} + \frac{s}{T^{n-1}} + \frac{s^2}{2! T^{n-2}} + \ldots + \frac{s^n}{n!} + \frac{s^{n+1} T}{(n + 1)!} + \frac{s^{n+2} T^2}{(n + 2)!} + \ldots}. \]

Since \( s^{n+k} T^k / (n+k)! \to \infty \) as \( T \to \infty \) for \( k = 1, 2, 3, \ldots \), we conclude that
\[ \frac{T^n}{e^{sT}} \to 0 \quad \text{as } T \to \infty. \]
for \( s > 0 \), regardless of the magnitude of \( n \). So if the integral in the second term on the right-hand side converges, its limit would be \( \mathcal{L}\{t^{n-1}\} \), which in turn would imply that the integral on the left-hand side converges to what would be \( \mathcal{L}\{t^n\} \). The end result is the recursion formula

\[
\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\} \quad (s > 0)
\]

provided \( \mathcal{L}\{t^{n-1}\} \) exists.

Substituting \( n = 1 \) into (10.2.2) gives

\[
\mathcal{L}\{t^1\} = \frac{1}{s} \mathcal{L}\{t^0\} = \frac{1}{s} \mathcal{L}\{1\}.
\]

Now \( \mathcal{L}\{1\} \) does exist since we determined in Example 10.2 that \( \mathcal{L}\{1\} = \frac{1}{s} \). Its existence implies the validity of formula (10.2.2) for \( n = 1 \) and thereby the existence of \( \mathcal{L}\{t^n\} \):

\[
\mathcal{L}\{t^n\} = \frac{1}{s^n}.
\]

The existence of \( \mathcal{L}\{t^n\} \) then validates (10.2.2) for \( n = 2 \); therefore,

\[
\mathcal{L}\{t^2\} = \frac{2}{s} \mathcal{L}\{t^1\} = \frac{2}{s} \cdot \frac{1}{s} = \frac{2}{s^2}.
\]

Continuing in this manner, we obtain

\[
\begin{align*}
n = 3 : & \quad \mathcal{L}\{t^3\} = \frac{3}{s} \mathcal{L}\{t^2\} = \frac{3}{s} \cdot \frac{2}{s} = \frac{3 \cdot 2}{s^3} \\
n = 4 : & \quad \mathcal{L}\{t^4\} = \frac{4}{s} \mathcal{L}\{t^3\} = \frac{4}{s} \cdot \frac{3 \cdot 2}{s} = \frac{4 \cdot 3 \cdot 2}{s^4} \\
n = 5 : & \quad \mathcal{L}\{t^5\} = \frac{5}{s} \mathcal{L}\{t^4\} = \frac{5}{s} \cdot \frac{4 \cdot 3 \cdot 2}{s} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{s^5}
\end{align*}
\]

and so on. The pattern indicates that

\[
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (s > 0)
\]

for every positive integer \( n \). Certainly, we have established that the formula is true for \( n = 1, 2, 3, 4, 5 \). Note that it is also true for \( n = 0 \) with the understanding that \( 0! = 1 \).

What about for \( n > 5 \)? Suppose (10.2.2) were true for an integer \( n = k \); that is,

\[
\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}}.
\]

The previous cases indicate that it then should be true for the next value of \( n \), to wit, \( n = k + 1 \). Let’s check this. By recursion formula (10.2.2)

\[
\mathcal{L}\{t^{k+1}\} = \frac{k + 1}{s} \mathcal{L}\{t^k\}.
\]

\[\text{Another way to show this limit is with L'Hôpital's Rule.}\]
10.2. FINDING LAPLACE TRANSFORMS WITH THE DEFINITION

This and (10.2.4) yields

\[ L \{ t^{k+1} \} = \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}} = \frac{(k+1)!}{s^{k+2}}. \]

So we have established that it is impossible for formula (10.2.3) to be true for some integer but not for the next one. The validity of the formula for \( n = 1, 2, 3, 4, \) and 5 has already been established. Since it is valid for \( n = 5, \) it must be valid for \( n = 6. \) Its validity for \( n = 6 \) implies its validity for \( n = 7, \) which in turn implies its validity for \( n = 8. \) This domino effect makes it valid for all positive integers. Some of you may recognize that we have established the validity of formula (10.2.3) for all positive integers by using a method of proof known as **mathematical induction**.

**Example 10.5.** Find the Laplace transform of \( L \{ \sin bt \} \), where \( b \) is a real constant.

**Solution.** By the definition of the Laplace Transform (10.1.12), for \( b \neq 0, \)

\[ L \{ \sin bt \} = \lim_{T \to \infty} \int_0^T e^{-st} \sin bt \, dt. \]

Applying the integration-by-parts formula twice to the integral, we find that

\[ \left(1 + \frac{b^2}{s^2}\right) \int_0^T e^{-st} \sin bt \, dt = -\frac{1}{s} e^{-sT} \sin bT + \frac{b}{s} e^{-sT} \cos bT + \frac{b}{s^2} \]

for \( s \neq 0. \) For \( s > 0, \) \( e^{-sT} \to 0 \) as \( T \to \infty. \) Since

\[ \left| e^{-sT} \sin bT \right| = e^{-sT} |\sin bT| \leq e^{-sT}, \]

we see that \( e^{-sT} \sin bT \to 0 \) as \( T \to \infty. \) Likewise \( e^{-sT} \cos bT \to 0 \) as \( T \to \infty. \) We conclude that

\[ \left(1 + \frac{b^2}{s^2}\right) L \{ \sin bt \} = \frac{b}{s^2}. \]

Solving for \( L \{ \sin bt \} \) gives

\[ L \{ \sin bt \} = \frac{b}{s^2 + b^2} \]

for \( s > 0. \) Note that this formula is valid for \( b = 0; \) so we can remove the stipulation that \( b \neq 0. \)

**Example 10.6.** Find the Laplace transform of the Dirac delta function at \( b \) for \( b \geq 0. \)

**Solution.** Once again by (10.1.12),

\[ L \{ \delta(t - b) \} = \int_0^\infty e^{-st} \delta(t - b) \, dt. \]

By the sifting property of the Dirac delta function,

\[ L \{ \delta(t - b) \} = \lim_{T \to \infty} \int_0^T e^{-st} \delta(t - b) \, dt = \lim_{T \to \infty} e^{-sb} u(T - b). \]
Since \( u(T - b) = 1 \) for \( T > b \),

\[
\mathcal{L}\{\delta(t - b)\} = e^{-bs}.
\]

In the next section, Table 10.1 lists all of the Laplace transforms that we worked out in this section. It also includes the transform of the cosine function:

\[
\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}
\]

for \( s > 0 \). This can be verified using Definition 10.1.

### 10.3 Linearity of the Laplace Transform

Even though Laplace transforms can be computed with a CAS or looked up in a variety of different handbooks or even in the appendices of many engineering textbooks, the transforms of some of the basic functions should be memorized. Some of these are listed in Table 10.1.

#### Table 10.1: Basic Laplace Transforms

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( t^n ), ( n = 0, 1, 2, \ldots )</td>
<td>( \frac{n!}{s^{n+1}} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( e^{bt} )</td>
<td>( \frac{1}{s - b} ), ( s &gt; b )</td>
</tr>
<tr>
<td>( \sin bt )</td>
<td>( \frac{b}{s^2 + b^2} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( \cos bt )</td>
<td>( \frac{s}{s^2 + b^2} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( u(t - b), \ b \geq 0 )</td>
<td>( \frac{e^{-bs}}{s} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( \delta(t - b), \ b \geq 0 )</td>
<td>( e^{-bs} )</td>
</tr>
</tbody>
</table>

Observe that the Laplace transforms of \( \sin bt \) and \( \cos bt \) are the same except for the numerators. The sentence

*Some birds can swim.*
10.3. LINEARITY OF THE LAPLACE TRANSFORM

can aid in remembering that the numerator of \(\sin bt\) is \(b\) whereas the numerator of \(\cos bt\) is \(s\).

Can we use this table to find the transform of a function that is a combination of some of the functions listed in it? It turns out that the answer is yes for special combinations known as linear combinations. A linear combination of the functions \(f_1, f_2, \cdots, f_n\) is a function of the form

\[
c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,
\]

where \(c_1, c_2, \cdots, c_n\) are constants. In integral calculus, we learned that the integration of a linear combination of integrable functions can be carried out by first integrating each of the functions and then by taking the same linear combination of the results; that is to say, in the case of \(n = \begin{bmatrix} 2 \end{bmatrix}:\)

\[
\int_a^b (c_1 f_1(t) + c_2 f_2(t)) \, dt = c_1 \int_a^b f_1(t) \, dt + c_2 \int_a^b f_2(t) \, dt.
\]

This property of integration is known as linearity. Since Laplace transforms are improper integrals, they inherit this linearity property, as we now state.

**Linearity of the Laplace Transform**

**Theorem 10.1.** Let \(f_1\) and \(f_2\) be functions whose Laplace transforms exist for \(s > b_1\) and \(s > b_2\), respectively. Let \(b\) be the larger of \(b_1\) and \(b_2\). Then for \(s > b\),

\[
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}, \tag{10.3.1}
\]

where \(c_1\) and \(c_2\) are any constants.

**Proof.** To show this, we merely use the linearity property of integration: For all \(T \geq 0,\)

\[
\int_0^T e^{-st} [c_1 f_1(t) + c_2 f_2(t)] \, dt = c_1 \int_0^T e^{-st} f_1(t) \, dt + c_2 \int_0^T e^{-st} f_2(t) \, dt.
\]

Then letting \(T \to \infty\), we obtain (10.3.1). \(\square\)

Special versions of (10.3.1) are

(a) \(\mathcal{L}\{f_1(t) + f_2(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\}\)

(b) \(\mathcal{L}\{c_1 f_1(t)\} = c_1 \mathcal{L}\{f_1(t)\}\)

Obviously, (a) follows from setting both constants in (10.3.1) equal to 1, while (b) follows from setting \(c_2\) equal to zero. So we have just showed that (10.3.1) implies (a) and (b). You should convince yourself that the converse is also true: (a) and (b) taken together imply (10.3.1). Therefore we can say that properties (a) and (b) are equivalent.
to property (10.3.1). An operator $O$, such as $L$ or $d/dt$ or anything else for that matter, with the property
\[ O(c_1 f_1(t) + c_2 f_2(t)) = c_1 O(f_1(t)) + c_2 O(f_2(t)) \]
for all functions $f_1$ and $f_2$ in its domain and all constants $c_1$ and $c_2$ is called a linear operator. So stating that $L$ is a linear operator is equivalent to stating that it satisfies property (10.3.1). We leave it to the reader to show that (10.3.1) can be extended to a linear combination of more than two functions:

\[ L\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} + \cdots + c_n L\{f_n(t)\}. \] (10.3.2)

Let’s demonstrate the use of the linearity property in determining Laplace transforms of some linear combinations of the functions listed in Table 10.1.

**Example 10.7.** Find $L\{t^3 + 2e^{5t}\}$.

**Solution.** By (10.3.1), or (10.3.2) with $n = 2$,
\[ L\{t^3 + 2e^{5t}\} = L\{t^3\} + 2L\{e^{5t}\}. \]
By Table 10.1, $L\{t^3\} = \frac{3!}{s^4}$ for $s > 0$ and $L\{e^{5t}\} = \frac{1}{s-5}$ for $s > 5$. Therefore, for $s > 5$,
\[ L\{t^3 + 2e^{5t}\} = \frac{3!}{s^4} + 2 \cdot \frac{1}{s-5} = \frac{6}{s^4} + \frac{2}{s-5}. \]

**Example 10.8.** Find the Laplace transform of the function
\[ f(t) = 5e^{-3t} - \frac{1}{2} \sin 6t + \pi. \]

**Solution.** It follows from (10.3.2) with $n = 3$ and Table 10.1 that
\[ L\{f(t)\} = 5L\{e^{-3t}\} - \frac{1}{2} L\{\sin 6t\} + L\{\pi \cdot 1\} \]
\[ = 5 \cdot \frac{1}{s-(-3)} - \frac{1}{2} \cdot \frac{6}{s^2 + 6^2} + \pi \cdot 1 = \frac{5}{s+3} - \frac{3}{s^2 + 36} + \frac{\pi}{s} \]
for $s > 0$.

### 10.4 First-Order Linear Equations

Recall that the primary objective of this chapter is to develop a method for solving linear differential equations using the Laplace transform, so we must bear this in mind as we search for more of its properties. Recall that we began this chapter with the initial value problem (10.1.1), namely,
\[ \frac{dx}{dt} + 2x = 3e^t, \quad x(0) = 1. \]
In our attempt to discover another way of solving (10.1.1), one that did not involve integrating factors, we stumbled upon the idea of the Laplace transform. Now that we have formally defined the Laplace transform, discovered that it is a linear operator, and found the transform of some commonly occurring functions, let’s retrace the steps that led to the solution of (10.1.1) and tidy them up so as to streamline the procedure of using Laplace transforms to find solutions. We began by tinkering with the differential equation in (10.1.1). This eventually resulted in equation (10.1.6). Then we took the limit of both of its sides as $T \to \infty$. Admittedly all of this was done in a clumsy, roundabout way; however, it is really more straightforward than it seems. Observe that the process of obtaining (10.1.6) and then taking its limit amounts to nothing more than taking the Laplace transform of both sides of the differential equation

$$L\{x' + 2x\} = L\{3e^t\}$$

and then using the linearity property

$$L\{x'\} + 2L\{x\} = 3L\{e^t\}.$$  

From Table 10.1, $L\{e^t\} = 1/(s-1)$; thus,

$$L\{x'\} + 2L\{x\} = \frac{3}{s-1}. \quad (10.4.1)$$

If $L\{x\}$ were the only unknown in the equation, then we could solve for it which would give us the transform of the solution. However, the presence of $L\{x'\}$ prevents this—unless there is a formula that relates $L\{x'\}$ to $L\{x\}$; if this were the case, then we could use it to convert the previous equation to one involving only $L\{x\}$. So the utility of Laplace transforms for solving differential equations depends on the existence of such a formula. Let’s look into this.

By Definition 10.1, the Laplace transform of the derivative of a function $f$ is

$$L\{f'(t)\} = \lim_{T \to \infty} \int_0^T e^{-st} f'(t) \, dt.$$  

Integrating by parts with $u = e^{-st}$, $dv = f'(t) \, dt$, $du = -se^{-st} \, dt$, and $v = f(t)$, we obtain

$$\int_0^T e^{-st} f'(t) \, dt = e^{-st} f(t)|_0^T + s \int_0^T e^{-st} f(t) \, dt \quad (10.4.2)$$

$$= e^{-sT} f(T) - f(0) + s \int_0^T e^{-st} f(t) \, dt.$$  

For $L\{f'\}$ to exist, the right-hand side must converge as $T \to \infty$. Now the last term converges to $sL\{f\}$ if $L\{f\}$ exists. There is no problem with the second term since it is a constant. But what about the first term? If you recall, we faced this same problem before in solving the initial value problem (10.1.1). There we assumed at the outset that its solution $x(t)$ would behave in the following manner: $e^{-st} x(t) \to 0$ as $t \to \infty$. After we found it, we verified that this was indeed the case for $s > 1$. This
behavior was also used in Example 10.4 to find the Laplace transform of \( f(t) = t^n \); there we proved for any fixed \( s > 0 \) that \( e^{-st} f(t) = e^{-st} t^n \to 0 \) as \( t \to \infty \). Let’s define a class of functions whose long-term behavior for any fixed value \( s > \gamma \) will turn out to be

\[
e^{-st} f(t) \to 0 \quad \text{as} \quad t \to \infty ,
\]

where the value of \( \gamma \) depends on the particular function \( f \).

### Exponential Order

**Definition 10.2.** A function \( f(t) \) is said to be of **exponential order** \( \gamma \) as \( t \to \infty \) if there are constants \( \gamma, M, \) and \( t_1 \) such that

\[
|f(t)| \leq M e^{\gamma t} \quad \text{for all} \quad t \geq t_1.
\]

We can always choose the value of \( t_1 \) to be nonnegative. At times, for the sake of brevity, we will say that \( f(t) \) is of exponential order—omitting the constant “\( \gamma \)” and the phrase “as \( t \to \infty \).” A concise way of writing this is

\[
f(t) = O(e^{\gamma t}). \quad \text{(10.4.5)}
\]

We say that \( f(t) \) is of the **order of** \( e^{\gamma t} \). The use of “\( O \)” in this way is called **big-oh notation**—so we can also say that “\( f(t) \) is big oh of \( e^{\gamma t} \).” In particular, when \( \gamma = 0 \), (10.4.4) becomes \( f(t) = O(e^{0t}) = O(1) \). Consequently,

\[
f(t) = O(1)
\]

will mean that \( |f(t)| \leq M \) for all \( t \geq t_1 \). This is a succinct way to say \( f(t) \) is bounded on the interval \([t_1, \infty)\) for some \( t_1 \geq 0 \).

Let’s look at some examples to become better acquainted with the notation and proficient at recognizing functions of exponential order. After that we can continue with our search for a formula for the Laplace transform of the derivative.

**Example 10.9.** Let \( f(t) = t^n \), for all \( t \geq 0 \), where \( n = 0, 1, 2, 3, \ldots \). From the Maclaurin series

\[
e^t = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots,
\]

it follows that \( e^t > t^n/n! \) and so

\[
|f(t)| = t^n \leq n! e^t \quad \text{for all} \quad t \geq 0.
\]

In the notation of (10.4.4), \( M = n! \), \( \gamma = 1 \), and \( t_1 = 0 \). This shows that \( f(t) = t^n \) is of exponential order 1. Equivalently, in terms of the big-oh notation, \( t^n = O(e^t) \). Actually, \( t^n \) is of exponential order \( \gamma \) for any \( \gamma > 0 \). To see this, let \( \gamma \) be any fixed positive number, no matter how small or large. While working through Example 10.4, we proved that \( e^{-\gamma t} t^n \to 0 \) as \( t \to \infty \). Since \( e^{-\gamma t} t^n \) is a continuous function, this
implies that \( e^{-yt}t^n \) is bounded. In other words, there exists a constant \( K > 0 \) such that
\[
|e^{-yt}t^n| \leq K \text{ for all } t \geq 0.
\]
Thus,
\[
|t^n| \leq Ke^{yt}
\]
for all \( t \geq 0 \), which is the same as saying that
\[
t^n = O\left(e^{yt}\right)
\]
for any positive number \( y \).

**Example 10.10.** The unit step function at \( b \), \( u(t - b) \), is of exponential order 0 as
\[
|u(t - b)| \leq 1 = 1 \cdot e^{0t}.
\]
Equivalently, we can write
\[
u(t - b) = O(1)
\]
as \( t \to \infty \).

**Example 10.11.** Consider the function
\[
g(t) = -6te^{3t/2} \sin t
\]
for \( t \geq 0 \). As \(-1 \leq \sin t \leq 1\) for all \( t \) and \( t < e^t \) for all \( t \geq 0 \),
\[
|g(t)| \leq 6te^{3t/2} \leq 6e^t e^{3t/2} = 6e^{5t/2}
\]
for \( t \geq 0 \). Thus, \( g \) is of exponential order 2.5.

**Example 10.12.** The function
\[
h(t) = e^{t^2}
\]
is not of exponential order. For if it were, then constants \( M, \gamma, \) and \( t_1 \) would exist such that
\[
|h(t)| = e^{t^2} \leq Me^{\gamma t} \text{ for all } t \geq t_1.
\]
Or, simply \( e^{t^2-\gamma t} \leq M \), for all \( t \geq t_1 \). But this contradicts that \( e^{(t-\gamma)} \to \infty \) as \( t \to \infty \).

**Example 10.13.** As a final example, consider the function
\[
p(t) = \frac{5}{\sqrt{t}}.
\]
Note that \( p(t) \to \infty \) as \( t \to 0^+ \). So the function is unbounded on the interval \((0, \infty)\). However, on \([1, \infty)\), \( 0 < p(t) \leq 5 \) as \( p \) is a positive, decreasing function and \( p(1) = 5 \). So we can certainly say that \( |p(t)| \leq 5 \), for all \( t \geq 1 \). Thus, by the remarks following (10.4.5), \( p(t) = O(1) \) as \( t \to \infty \).
Now that we know what the class of functions of exponential order are and looked at some examples, let’s verify that the members of this class display the long-term behavior that was claimed earlier in (10.4.3). Suppose \( f(t) = O(e^{\gamma t}) \) as \( t \to \infty \). This means according to (10.4.4) that there are constants \( M > 0 \) and \( t_1 \geq 0 \) such that

\[
-M e^{\gamma t} \leq f(t) \leq M e^{\gamma t} \quad \text{for all } t \geq t_1.
\]

Multiplying both inequalities by \( e^{-st} \), we have

\[
-M e^{-(s-\gamma)t} \leq e^{-st} f(t) \leq M e^{-(s-\gamma)t}.
\]

If \( s > \gamma, \pm M e^{-(s-\gamma)t} \to 0 \) as \( t \to \infty \). Since the middle quantity, \( e^{-st} f(t) \), is sandwiched between two other quantities that approach 0 as \( t \to \infty \), it too must approach 0. Thus, we have the result:

**Long-Term Behavior of Functions of Exponential Order**

**Theorem 10.2.** If \( f(t) = O(e^{\gamma t}) \) as \( t \to \infty \), then for \( s > \gamma \),

\[
\lim_{t \to \infty} e^{-st} f(t) = 0. \tag{10.4.6}
\]

At last we have all the necessary pieces in place to derive a formula relating the Laplace transform of the derivative of a function to the transform of the function itself.

**Transform of the First Derivative**

**Theorem 10.3.** Suppose a function \( f(t) \) is differentiable on \([0, \infty)\) and of exponential order \( \gamma \), its Laplace transform exists, and its derivative \( f'(t) \) is integrable on \([0, T]\) for every \( T \geq 0 \). Then the Laplace transform of its derivative exists for all \( s > \gamma \) and is given by the formula

\[
\mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0). \tag{10.4.7}
\]

**Proof.** For every \( T \geq 0 \), \( e^{-st} f'(t) \) is integrable on \([0, T]\) since both of its factors are integrable on \([0, T]\). This, along with the differentiability of \( e^{-st} \) and \( f(t) \), allows us to integrate \( e^{-st} f'(t) \) by parts, which was done earlier and resulted in (10.4.2):

\[
\int_0^T e^{-st} f'(t) \, dt = e^{-sT} f(T) - f(0) + s \int_0^T e^{-st} f(t) \, dt.
\]

It follows from (10.4.6) that for \( s > \gamma, e^{-sT} f(T) \to 0 \) as \( T \to \infty \). By the hypotheses, the Laplace transform of \( f \) exists; hence, \( f_0^T e^{-st} f(t) \, dt \to \mathcal{L}\{f(t)\} \) as \( T \to \infty \). 

---

8If the functions \( f \) and \( g \) are integrable on an interval \([a, b]\), then their product \( fg \) is also integrable on \([a, b]\). For a proof of this, see any advanced calculus or real analysis book, such as Ross [46, p. 197].

9See Ross [46, p. 200].
Example 10.14. Find the Laplace transform of \( \cos bt \), where \( b \) is a real constant, given that the Laplace transform of \( \sin bt \) is

\[
\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}
\]

Solution. Let \( f(t) = \sin bt \). Then, \( f'(t) = b \cos bt \) and \( f(0) = 0 \). Then (10.4.7) and the result of Example 10.5 yields

\[
\mathcal{L}\{b \cos bt\} = s \mathcal{L}\{\sin bt\} - 0 = s \frac{b}{s^2 + b^2}.
\]

By linearity,

\[
\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}.
\]

Example 10.15. Find \( \mathcal{L}\{\sin^2 bt\} \) given that \( \mathcal{L}\{\sin bt\} = b/(s^2 + b^2) \).

Solution. Let \( f(t) = \sin^2 bt \). Then \( f(0) = 0 \). Substituting into (10.4.7), we obtain

\[
\mathcal{L}\left\{\frac{d}{dt} \sin^2 bt\right\} = s \mathcal{L}\{\sin^2 bt\} - 0.
\]

Carrying out the differentiation yields

\[
\mathcal{L}\{\sin^2 bt\} = \frac{1}{s} \mathcal{L}\{2b(\sin bt)(\cos bt)\}.
\]

By the double-angle identity \( 2(\sin bt)(\cos bt) = \sin 2bt \). Hence,

\[
\mathcal{L}\{\sin^2 bt\} = \frac{1}{s} \mathcal{L}\{b \sin 2bt\} = \frac{b}{s} \cdot \frac{b}{s^2 + (2b)^2} = \frac{b^2}{s(s^2 + 4b^2)}.
\]

Now that we have a formula relating the transform of a function to the transform of its derivative, namely (10.4.7), let’s go back to (10.4.1) to continue retracing the steps that led to the solution of the initial value problem (10.1.1). Recall it is our intention to streamline the roundabout way in which we used Laplace transforms to solve (10.1.1). This will be accomplished through the \( \mathcal{L} \) operator. As

\[
\mathcal{L}\{x'(t)\} = s \mathcal{L}\{x(t)\} - x(0)
\]

and

\[ x(0) = 1, \]

(10.4.1) becomes

\[
s \mathcal{L}\{x(t)\} - 1 + 2 \mathcal{L}\{x(t)\} = \frac{3}{s - 1}.
\]
It is here that we should point out one of the advantages in using the Laplace transform to solve initial value problems. Note that $L$ has transformed (10.1.1), the differential equation and the initial condition, into an algebraic equation:

$$\begin{cases} x' + 2x = 3e^t \\ x(0) = 1 \end{cases} \overset{L}{\longrightarrow} sL\{x(t)\} - 1 + 2L\{x(t)\} = \frac{3}{s - 1}. $$

Solving the algebraic equation for the unknown $L\{x(t)\}$, we get

$$(s + 2)L\{x(t)\} = \frac{3}{s - 1} + 1 = \frac{s + 2}{s - 1}$$

and so

$L\{x(t)\} = \frac{1}{s - 1}.$

Recall this is (10.1.10) on page 316. Of course, this is the transform of the solution of (10.1.1), not the solution. Let’s look for a continuous function with this transform. From Table 10.1, we see that $\frac{1}{s - 1} = L\{e^t\}$.

Thus,

$L\{x(t)\} = L\{e^t\}$.

From this we surmise that

$x(t) = e^t.$

Recall from earlier discussions that this is indeed the unique solution of the initial value problem (10.1.1).

When using transforms to solve an initial value problem, we will eventually arrive at the point where the transform of the solution $x(t)$ is expressed in terms of the transform of some other function, say $f(t)$. That is,

$L\{x(t)\} = L\{f(t)\}.$

(For example, we found $L\{x(t)\} = L\{e^t\}$ in the previous discussion.) So we ask if $x(t) = f(t)$? The answer to this is given by a special version of a result known as Lerch’s theorem. For continuous functions, it says that different continuous functions have different Laplace transforms. Therefore, there can only be one continuous function whose transform is equal to the transform of the solution.

### Uniqueness of Inverse Laplace Transforms

**Theorem 10.4 (Lerch’s theorem).** Suppose $f(t)$ and $g(t)$ are continuous on $[0, \infty)$ and of exponential order. If $L\{f(t)\} = L\{g(t)\}$ for all $s > \gamma$ for some $\gamma \geq 0$, then $f(t) = g(t)$ for all $t \geq 0$.

---

10 Other advantages will be pointed out later, such as the efficient manner in which the transform can handle discontinuous functions.

11 Mathias Lerch (1860–1922), born in Czechoslovakia, was a professor of mathematics at the University of Freiburg in Germany and at Masaryk University in Czechoslovakia. He published about 240 papers, mostly in analysis and number theory. Some of his results are important to the study of operators, such as the Laplace transform operator.
10.4. FIRST-ORDER LINEAR EQUATIONS

This version of Lerch’s theorem is stated and proved in Marsden [46, p. 390; p. 398 f.]. We do not prove it here because the proof depends on advanced results from real and complex analysis. A statement of Lerch’s theorem and where its proof can be found is given in Churchill’s *Operational Mathematics* [16, p. 14; p. 184].

**Example 10.16.** Let’s try out this method we are in the process of developing on

\[
\frac{dx}{dt} - 2x = 4, \quad x(0) = 0.
\]  

(10.4.8)

**Solution.** First take the Laplace transform of both sides of the above differential equation:

\[L\{x' - 2x\} = L\{4\}.
\]

Next use the initial condition and properties of the transform to express the equation obtained in part (a) in terms of the transform of the solution. First linearity gives

\[L\{x'\} - 2L\{x\} = 4L\{1\}.
\]

Then the transform of the constant function “1” and (10.4.7) yields

\[sL\{x\} - x(0) - 2L\{x\} = 4 \cdot \frac{1}{s}.
\]

Since the initial condition is \(x(0) = 0\), this simplifies to

\[sL\{x\} - 2L\{x\} = \frac{4}{s}.
\]

Solving this equation for \(L\{x(t)\}\), we obtain the transform of the solution \(x(t)\):

\[L\{x(t)\} = \frac{4}{s(s - 2)}.
\]  

(10.4.9)

Now that we have \(L\{x(t)\}\), we can obtain the solution \(x(t)\) itself by finding a continuous function with this particular transform. But the right-hand side of (10.4.9) or anything resembling it does not appear in Table 10.1 on page 324. We could look for it in an engineering or math handbook that has a more extensive table of Laplace transforms. There we would find an entry for \([a(s - b)]^{-1}\). Even so, we can express (10.4.9) in terms of the functions listed in Table 10.1. We would run into this same problem if we had to integrate (10.4.9): in that case we would simply write it as a sum of two rational functions before integrating—in other words, we would first apply the method of partial fractions. Let’s try that here. The form of the partial fraction expansion of (10.4.9) is

\[
\frac{4}{s(s - 2)} = \frac{A}{s - 2} + \frac{B}{s}.
\]

To find \(A\) and \(B\), we first multiply by \(s(s-2)\) to clear the fractions of their denominators obtaining

\[As + B(s - 2) = 4.
\]
Then set \( s = 2 \) to find \( A \):
\[
s = 2 \quad \Rightarrow \quad 2A = 4 \quad \Rightarrow \quad A = 2.
\]
And \( s = 0 \) to find \( B \):
\[
s = 0 \quad \Rightarrow \quad -2B = 4 \quad \Rightarrow \quad B = -2.
\]
Therefore,
\[
\frac{4}{s(s-2)} = \frac{2}{s-2} - \frac{2}{s}
\]
and so
\[
\mathcal{L}\{x(t)\} = 2 \cdot \frac{1}{s-2} - 2 \cdot \frac{1}{s}.
\]
From Table 10.1 and the linearity of the transform, we see that
\[
2 \cdot \frac{1}{s-2} - 2 \cdot \frac{1}{s} = 2\mathcal{L}\{e^{2t}\} - 2\mathcal{L}\{1\} = \mathcal{L}\{2e^{2t} - 2\}.
\]
Since \( \mathcal{L}\{x(t)\} = \mathcal{L}\{2e^{2t} - 2\} \), we conclude from Theorem 10.4 that the solution of the initial value problem (10.4.8) is
\[
x(t) = 2e^{2t} - 2.
\]

An examination of how we solved the initial value problems (10.1.1) and (10.4.8) provides the steps for solving any linear initial value problem with Laplace Transforms.

**The Method of Laplace Transforms**

To solve an initial value problem:

1. Take the Laplace transform of both sides of the differential equation.
2. Use the initial condition and properties of the transform to express the algebraic equation obtained in Step 1 in terms of the transform of the solution.
3. Solve the equation obtained in Step 2 for the transform of the solution.
4. Determine the solution by finding a continuous function with a transform the same as the one obtained in Step 3.

Before we work out any more examples, let’s point out the assumptions which are unsaid but implicit when we employ the above method. In Step 1, we assume the existence of a function \( x(t) \) that is continuous on \([0, \infty)\) and satisfies the first-order differential equation and accompanying initial condition. This is a valid assumption since for any pair of continuous functions \( p \) and \( q \) on \([0, \infty)\) there is a unique continuous solution \( x(t) \) of
\[
x'' + p(t)x = q(t), \quad x(0) = x_0
\]
10.4. FIRST-ORDER LINEAR EQUATIONS

335

on \([0, \infty)\). (See the existence and uniqueness theorem in Chapter 5.) It follows from the continuity of the right-hand side of

\[ x'(t) = q(t) - p(t)x(t) \]

that the derivative \(x'(t)\) is continuous on \([0, \infty)\). Consequently, \(x'(t)\) is integrable on \([0, T]\) for every \(T \geq 0\). So some of the hypotheses of Theorem 10.3 are automatically fulfilled. This still leaves open the question of the existence of the Laplace transform of \(x(t)\) and whether it is of exponential order. What we do in practice is to go ahead with the method of Laplace transforms assuming that all the hypotheses are satisfied. If Steps 3 and 4 can be carried out, then we can check if the result of Step 4 fulfills the hypotheses and is a solution.

In the next section we take a look at conditions that allow us to determine the existence of the Laplace transform of a function and avoid the tedium that comes from using Definition 10.1. But first we illustrate the method of Laplace transforms by solving a couple of initial value problems.

Example 10.17. Solve the initial value problem

\[ \frac{dx}{dt} - x = -t, \quad x(0) = 1. \quad (10.4.10) \]

Solution. First take the Laplace transform of both sides of the differential equation and apply the linearity property:

\[ \mathcal{L}\{x'\} - \mathcal{L}\{x\} = -\mathcal{L}\{t\}. \]

Then use Theorem 10.3, the entry for \(t^n\) with \(n = 1\) in Table 10.1, and the alternative notation \(X(s)\) for \(\mathcal{L}\{x(t)\}\) to get

\[ sX(s) - x(0) - X(s) = -\frac{1}{s^2}. \]

Next use the initial condition \(x(0) = 1\) and solve this equation for \(X(s)\):

\[ X(s) = \frac{1}{s - 1} - \frac{1}{(s - 1)s^2}. \]

Factor out \(1/(s - 1)\) and simplify:

\[ X(s) = \frac{1}{s - 1} \left[ 1 - \frac{1}{s^2} \right] = \frac{1}{s - 1} \left[ \frac{s^2 - 1}{s^2} \right] = \frac{s + 1}{s^2} = \frac{1}{s} + \frac{1}{s^2}. \]

Thus, the transform of the solution \(x(t)\) of the initial value problem (10.4.10) is

\[ \mathcal{L}\{x(t)\} = \frac{1}{s} + \frac{1}{s^2}. \]

To find the solution \(x(t)\) itself, we see from Table 10.1 that

\[ \frac{1}{s} + \frac{1}{s^2} = \mathcal{L}\{1\} + \mathcal{L}\{t\}. \]
Consequently,
\[ \mathcal{L}\{x(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{t\} = \mathcal{L}\{1 + t\}. \]

Therefore, by Theorem 10.4,
\[ x(t) = 1 + t. \]

**Example 10.18.** Solve
\[ x' + 4x = 2 - 25 \cos 3t, \quad x(0) = 2. \]

**Solution.** Taking the Laplace transform, we get
\[ \mathcal{L}\{x'\} + 4\mathcal{L}\{x\} = 2\mathcal{L}\{1\} - 25\mathcal{L}\{\cos 3t\}. \]

As in the previous example, use Table 10.1 and the transform of the first derivative property:
\[ sX(s) - x(0) + 4X(s) = 2 \cdot \frac{1}{s} - 25 \cdot \frac{s}{s^2 + 3^2}, \]

where \( X(s) = \mathcal{L}\{x(t)\} \). Solving for \( X(s) \), we get
\[ X(s) = \frac{2}{s + 4} + \frac{2}{s(s + 4)} - \frac{25s}{(s + 4)(s^2 + 9)}. \]

Now we need to determine the solution by finding a continuous function with this same transform. Unfortunately, the second and third terms are not listed in Table 10.1. However, we can still use the table by expressing these terms as sums of partial fractions. First let’s expand the second term by expressing it as the sum
\[ \frac{2}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}. \]

To obtain the values of the coefficients, we multiply both of its sides by \( s(s + 4) \):
\[ A(s + 4) + Bs = 2. \]

Now we can readily obtain the values of \( A \) and \( B \) from this equation by setting \( s = 0 \) and \( s = -4 \):
\[ s = 0 \quad \Rightarrow \quad 4A = 2 \quad \Rightarrow \quad A = \frac{1}{2}; \]
\[ s = -4 \quad \Rightarrow \quad -4B = 2 \quad \Rightarrow \quad B = -\frac{1}{2}. \]

Thus,
\[ \frac{2}{s(s + 4)} = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s + 4}. \quad (10.4.11) \]

The partial fraction expansion of the third term is
\[ \frac{25s}{(s + 4)(s^2 + 9)} = \frac{C}{s + 4} + \frac{Ds + E}{s^2 + 9}. \]
Multiplying by \((s + 4)(s^2 + 9)\) gives
\[
C(s^2 + 9) + (Ds + E)(s + 4) = 25s.
\]
Successively setting \(s\) equal to \(-4, 0,\) and \(1\) gives
\[
s = -4 : \quad 25C = -100 \quad \Rightarrow \quad C = -4
\]
\[
s = 0 : \quad 9C + 4E = 0 \quad \Rightarrow \quad E = -\frac{9}{4}C = 9
\]
\[
s = 1 : \quad 10C + 5(D + E) = 25 \quad \Rightarrow \quad D = 5 - 2C - E = 5 - 2(-4) - 9 = 4.
\]
Therefore,
\[
\frac{25s}{(s + 4)(s^2 + 9)} = -4 \cdot \frac{1}{s + 4} + 4 \cdot \frac{s + 9}{s^2 + 9}
\]
\[
= -4 \cdot \frac{1}{s + 4} + 4 \cdot \frac{s}{s^2 + 9} + 3 \cdot \frac{3}{s^2 + 9}.
\]
Now let’s substitute the two partial fraction expansions, (10.4.11) and (10.4.12), for the second and third terms of \(X(s)\) respectively and then simplify:
\[
X(s) = \frac{1}{2} \cdot \frac{1}{s} + \frac{11}{2} \cdot \frac{1}{s - (-4)} - 4 \cdot \frac{s}{s^2 + 3^2} - 3 \cdot \frac{3}{s^2 + 3^2}
\]
\[
= \frac{1}{2} \mathcal{L}\{1\} + \frac{11}{2} \mathcal{L}\{e^{-4t}\} - 4 \mathcal{L}\{\cos 3t\} - 3 \mathcal{L}\{\sin 3t\}.
\]
This is the same as
\[
\mathcal{L}\{x(t)\} = \mathcal{L}\left\{\frac{1}{2} + \frac{11}{2} e^{-4t} - 4 \cos 3t - 3 \sin 3t\right\}
\]
by the linearity property. Finally, from the uniqueness of transforms property, we conclude that the solution is
\[
x(t) = \frac{1}{2} + \frac{11}{2} e^{-4t} - 4 \cos 3t - 3 \sin 3t.
\]

### 10.5 Existence of the Laplace Transform

In this section we state and prove some results guaranteeing the existence of Laplace transforms for certain types of functions. The following theorem is one of those results.

**Existence of the Laplace Transform I**

**Theorem 10.5.** If a function \(f(t)\) is bounded and integrable on every finite interval \([0, T]\) and is of exponential order \(\gamma\), then its Laplace transform exists for \(s > \gamma\).
Proof. To prove existence, we have to show the improper integral \( \int_0^\infty e^{-st} f(t) \, dt \) converges for \( s > \gamma \). That is, we must prove \( \int_0^T e^{-st} f(t) \, dt \) exists for every \( T > 0 \) and \( s > \gamma \) and that it approaches a finite limit as \( T \to \infty \). Clearly the integral exists on \([0, T]\) since \( e^{-st} f(t) \) is bounded and integrable on \([0, T]\). And as \( f(t) \) is of exponential order \( \gamma \), there are constants \( M \) and \( t_1 \) such that \( |f(t)| \leq Me^{\gamma t} \) for all \( t \geq t_1 \). In fact, since \( f(t) \) is bounded on \([0, T]\) we can even say that

\[
|f(t)| \leq Me^{\gamma t}
\]

for all \( t \geq 0 \) by increasing the value of \( M \) if necessary. Now we can address the question of the convergence of \( \int_0^\infty e^{-st} f(t) \, dt \) for \( s > \gamma \). Observe that the integrand is dominated by the exponential function \( Me^{-(s-\gamma)t} \) for \( t \geq 0 \) since

\[
|e^{-st} f(t)| = e^{-st} |f(t)| \leq e^{-st} Me^{\gamma t} \leq Me^{-(s-\gamma)t}.
\]

(10.5.1)

Then as \( s > \gamma \),

\[
\lim_{T \to \infty} \int_0^T Me^{-(s-\gamma)t} \, dt = \lim_{T \to \infty} \left[ \frac{M}{-(s-\gamma)} e^{-(s-\gamma)t} \right]_0^T = \frac{M}{s-\gamma}.
\]

(10.5.2)

By a well-known comparison test for improper integrals, \( \int_0^\infty e^{-st} f(t) \, dt \) converges. That is, \( \mathcal{L}\{f(t)\} \) exists for \( s > \gamma \).

Example 10.19. In Example 10.11, we showed

\[
g(t) = -6te^{3t/2} \sin t
\]

is of exponential order 2.5. Since \( g \) is continuous everywhere, it is integrable on every finite interval \([0, T]\). Therefore, the theorem allows us to conclude that \( \mathcal{L}\{g(t)\} \) exists for \( s > 2.5 \) without having to go through the trouble of finding its transform. Sometimes this is all we need to know.

Example 10.20. The function \( tu(t-b) \) for \( b > 0 \) is discontinuous at \( t = b \); nevertheless, it is integrable on every finite interval \([0, T]\). Since for \( t > 0 \),

\[
|tu(t-b)| \leq t < e^t,
\]

the function is of exponential order 1. Therefore, its transform exists for \( s > 1 \).

The proof of Theorem 10.5 suggests:

---

12 The product of two integrable functions is integrable. See Ross [46, p. 197] or Fulks [25, p. 120].
13 If \( g \) and \( h \) are integrable on \([t_1, T]\) for every \( T > t_1 \) and if \( |g(t)| \leq h(t) \) for all \( t \geq t_1 \), then \( \int_{t_1}^T g(t) \, dt \) converges if \( \int_{t_1}^\infty h(t) \, dt \) converges. For more information, see a real analysis textbook, such as Bartle [4, p. 262] or Fulks [25, p. 408, p. 472]. Weaker versions of the comparison test are stated in elementary calculus textbooks, such as Stewart [53, p. 572] or Thomas and Finney [55, p. 525].
10.5. EXISTENCE OF THE LAPLACE TRANSFORM

Long-Term Behavior of Laplace Transforms

**Corollary 10.1.** If a function \( f(t) \) satisfies the hypotheses of Theorem 10.5, then

\[
\lim_{s \to \infty} \mathcal{L}\{f\}(s) = 0.
\]

(10.5.3)

**Proof.** We see from (10.5.1) and (10.5.2) that

\[
|\mathcal{L}\{f\}(s)| = \left| \int_0^\infty e^{-st} f(t) \, dt \right| \leq \int_0^\infty |e^{-st} f(t)| \, dt \leq \frac{M}{s - \gamma}
\]

for \( s > \gamma \). Consequently, the transform is bounded:

\[
0 \leq |\mathcal{L}\{f\}(s)| \leq \frac{M}{s - \gamma}.
\]

Since the lower bound is 0 and the upper bound approaches 0 as \( t \to \infty \), (10.5.3) follows.

Take for example the function

\[
F(s) = \frac{s}{s - 1}.
\]

This result implies that there is no bounded, integrable function of exponential order that has \( F(s) \) as its Laplace transform. And that is because

\[
\lim_{s \to \infty} F(s) = \lim_{s \to \infty} \frac{s}{s - 1} = 1.
\]

Corollary 10.1 will be put to good use later on in Section 10.8.1 to help solve differential equations with variable coefficients.

10.5.1 Piecewise Continuous Functions

Continuous functions suffice for modeling many phenomena and situations. Nevertheless, other models require a broader class of functions. One useful class is the set of functions that are discontinuous at isolated points but whose Laplace transforms exist. In order to fully describe these functions, we must first understand what is meant by a jump discontinuity. A function \( f(t) \) has a **jump discontinuity** at \( t = \tau \) if it is discontinuous at \( \tau \) but the left-and right-hand limits

\[
\lim_{t \to \tau^-} f(t) \quad \text{and} \quad \lim_{t \to \tau^+} f(t)
\]

exist and are finite. If the point \( t = \tau \) is an endpoint of an interval, then we still say \( f(t) \) has a jump discontinuity at \( \tau \) if the appropriate one-sided limit exists and is finite.
Definition 10.3.

- A function \( f(t) \) is **piecewise continuous on a finite interval** \([a, b]\) if it is continuous at each point in \([a, b]\) except for at most a finite number of jump discontinuities.

- A function \( f(t) \) is **piecewise continuous on the infinite interval** \([0, \infty)\) if it is piecewise continuous on every finite interval \([0, T]\).

**Example 10.21.** The unit step function at \( b > 0 \) is continuous on \([0, \infty)\) except at \( b \). At this point, \( u(t - b) \) has a jump discontinuity since the following one-sided limits are finite:

\[
\lim_{t \to b^-} u(t - b) = 0 \quad \text{and} \quad \lim_{t \to b^+} u(t - b) = 1.
\]

Hence, \( u(t - b) \) is piecewise continuous on \([0, \infty)\).

**Existence of the Laplace Transform II**

**Corollary 10.2.** If a function \( f(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \gamma \), then its Laplace transform exists for \( s > \gamma \).

**Proof.** Let \([0, T]\) be any finite interval. It follows from the definition of piecewise continuity that \( f(t) \) is bounded and has at most a finite number of jump discontinuities on \([0, T]\). This implies \( f(t) \) is integrable on \([0, T]\).\(^{14}\) By Theorem 10.5, \( \mathcal{L}\{f(t)\} \) exists for \( s > \gamma \).

10.6 **Properties of the Laplace Transform**

We have already found two properties of the Laplace transform that are needed to solve initial value problems:

- the linearity property for transforms of linear combinations of functions:

\[
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}
\]

- the property for taking the transform of the first derivative of a function:

\[
\mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0).
\]

Now we will derive some other very useful properties. The first two of these involve shifting (translating) along the \( s \)-axis and the \( t \)-axis.

\(^{14}\)See Bartle and Sherbert [5, p. 252], Ross [46, p. 196], or Gaughan [26, p. 167].
10.6. PROPERTIES OF THE LAPLACE TRANSFORM

10.6.1 Shift in the s-Domain

Oftentimes we need the transform of a product of two functions, one of which is an exponential function while the other is a function whose Laplace transform we already have. In other words, the product has the form $e^{bt} f(t)$, where $b$ is a constant and $\mathcal{L}\{f\}$ is known or can easily be found. This particular product is important; one reason being that it may turn up as a solution of a linear equation. For example, you can check that $e^{5t} \sin t$ is a solution of $\frac{d}{dt} y - 5y = e^{2t} \cos t$.

Now that we have established the importance of knowing the transform of $e^{bt} f(t)$, let’s get to the business of finding it. First, we assume that $\mathcal{L}\{f\}$ exists and is equal to a function $F(s)$ for $s > a$. By (10.1.12),

$$\mathcal{L}\{e^{bt} f(t)\} = \int_0^\infty e^{-st} e^{bt} \, dt = \int_0^\infty e^{-(s-b)t} f(t) \, dt.$$ 

Since

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt,$$

it follows that

$$\mathcal{L}\{e^{bt} f(t)\} = F(s - b)$$

for $s - b > a$. In other words, when $f$ is multiplied by $e^{bt}$, its Laplace transform can be obtained by merely shifting the argument of the Laplace transform of $f$ by $b$ units. Since the shift takes place on the $s$-axis, we refer to this property as the shift in the $s$-domain. Shortly, we will take a look at a shift property that takes place on the $t$-axis.

**Shift in the s-Domain**

**Theorem 10.6.** If the Laplace transform of $f(t)$ exists and is equal to the function $\mathcal{L}\{f(t)\} = F(s)$ for $s > a$, then

$$\mathcal{L}\{e^{bt} f(t)\} = F(s - b)$$

for $s > a + b$.

**Example 10.22.** Find $\mathcal{L}\{e^{2t} \cos 3t\}$.

**Solution.** We begin with

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2 + 3^2} = F(s)$$

for $s > 0$. In the notation of (10.6.1), $a = 0$ and $b = 2$. Thus,

$$\mathcal{L}\{e^{2t} \cos 3t\} = F(s - 2) = \frac{s - 2}{(s - 2)^2 + 3^2}$$

for $s > a + b = 2$. 

Example 10.23. Find $\mathcal{L}\{e^{-t}t^5\}$.

Solution. Start with

$$\mathcal{L}\{t^5\} = \frac{5!}{s^6} = F(s).$$

Now set $b = -1$ in (10.6.1):

$$\mathcal{L}\{e^{-t}t^5\} = F(s + 1) = \frac{5!}{(s + 1)^6}$$

for $s > -1$.

Sometimes we use the shift property backwards as the next example illustrates.

Example 10.24. Find a function $f$ with the Laplace transform

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2 - s + 1}.$$

Solution. Although $1/(s^2 - s + 1)$ is not listed directly in Table 10.1, it is related to the $\sin bt$ entry. To see this, complete the square of its denominator to write it in the form $b/(s^2 + b^2)$:

$$\frac{1}{s^2 - s + 1} = \frac{1}{(s - \frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{(s - \frac{1}{2})^2 + (\sqrt{\frac{3}{2}})^2}.$$

From Table 10.1, we see

$$\frac{\sqrt{3}}{s^2 + (\sqrt{\frac{3}{2}})^2} = \mathcal{L}\{\sin \frac{\sqrt{3}}{2}t\}.$$

By the shift property (10.6.1),

$$\frac{\sqrt{3}}{(s - \frac{1}{2})^2 + (\sqrt{\frac{3}{2}})^2} = \mathcal{L}\{e^{\frac{1}{2}t} \sin \left(\frac{\sqrt{3}}{2}t\right)\}.$$

Thus,

$$f(t) = \frac{2}{\sqrt{3}} e^{\frac{1}{2}t} \sin \left(\frac{\sqrt{3}}{2}t\right).$$

10.6.2 Shift in the $t$-Domain

Abrupt changes occur in a variety of physical situations. Examples that we considered in Chapter 7 were:

- culling a certain number of animals from a herd,
• hitting a baseball with a bat, and
• depositing money to an account at regular intervals.

There are other examples which you will most likely study in the future, such as turning a switch on and off in an electrical circuit.

Situations involving abrupt changes such as those cited above are often modeled with linear nonhomogeneous differential equations where the nonhomogeneous terms involve functions with jump discontinuities. One such function is the unit step function \( u(t - b) \). Recall its Laplace transform is

\[
L\{u(t - b)\} = \frac{e^{-bs}}{s},
\]

where \( b \) is a non-negative constant. With this transform we can solve linear equations with nonhomogeneous terms that are constant multiples of \( u(t - b) \). But what about equations with more complicated nonhomogeneous terms, such as:

\[
f(t)u(t - b) = \begin{cases} 
0, & \text{if } t < b \\
f(t), & \text{if } t > b
\end{cases}
\] (10.6.2)

To answer this, let’s start off assuming the transform of \( f(t) \) exists for \( s > a \). That is, assume

\[
L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) \, dt
\]

exists for \( s > a \). Applying the definition of the Laplace transform to (10.6.2), we have

\[
L\{f(t)u(t - b)\} = \int_0^\infty e^{-st} f(t)u(t - b) \, dt = \int_b^\infty e^{-st} f(t) \, dt,
\]

which is the result of the integrand being equal to 0 when \( t < b \) and to \( e^{-st} f(t) \) when \( t > b \). Now move the lower limit of integration back to 0 with a change of variables in order to rewrite the integral as a Laplace transform. This is accomplished by letting \( \tau := t - b \) so that \( \tau = 0 \) when \( t = b \). Consequently,

\[
L\{f(t)u(t - b)\} = \int_0^\infty e^{-s(t+b)} f(\tau + b) \, d\tau = e^{-bs} \int_0^\infty e^{-st} f(t + b) \, dt.
\]

Note that the integral on the right-hand side is the transform of the function \( f(t + b) \). Thus, we have shown

\[
L\{f(t)u(t - b)\} = e^{-bs} L\{f(t + b)\}.
\]

We can use this to find other Laplace transforms. Since it involves shifting the argument of \( f \), the domain of which is on the \( t \)-axis, we call it the \textit{shift in the t-domain property}. Applying the foregoing steps to \( f(t - b)u(t - b) \), we obtain the alternate form

\[
L\{f(t - b)u(t - b)\} = e^{-bs} L\{f(t)\}.
\]

This form of the shift property is particularly useful when solving certain initial value problems.
**Shift in the t-Domain**

**Theorem 10.7.** Let \( b \geq 0 \). If the Laplace transform of \( f(t) \) exists and is equal to the function \( \mathcal{L}\{f(t)\}(s) \) for \( s > a \), then the Laplace transform of \( f(t)u(t - b) \) exists for \( s > a \) and is given by the shift formula

\[
\mathcal{L}\{f(t)u(t - b)\} = e^{-bs}\mathcal{L}\{f(t + b)\}.
\]

(10.6.3)

A variant of the shift formula suited for solving initial value problems is

\[
\mathcal{L}\{f(t - b)u(t - b)\} = e^{-bs}\mathcal{L}\{f(t)\}.
\]

(10.6.4)

**Example 10.25.** Find the Laplace transform of \( e^{2t}u(t - 1) \).

*Solution.* Using the notation of (10.6.3), \( f(t) = e^{2t} \) and \( b = 1 \). Thus,

\[
f(t + b) = f(t + 1) = e^{2(t+1)} = e^2 \cdot e^{2t}
\]

and so

\[
\mathcal{L}\{f(t + b)\} = e^2 \cdot \mathcal{L}\{e^{2t}\} = \frac{e^2}{s - 2}.
\]

Therefore, by (10.6.3)

\[
\mathcal{L}\{e^{2t}u(t - 1)\} = e^{-s} \cdot \frac{e^2}{s - 2} = \frac{e^{2-s}}{s - 2}
\]

**Example 10.26.** Find the Laplace transform of

\[
g(t) = \begin{cases} 
0, & \text{if } t < \pi \\
sin(0.5t), & \text{if } t > \pi.
\end{cases}
\]

*Solution.* The function \( g \) has a jump discontinuity at \( t = \pi \). In terms of the unit step function at \( \pi \), it is

\[
g(t) = \sin(0.5t) \cdot u(t - \pi).
\]

In terms of the notation of (10.6.3), \( f(t) = \sin(0.5t) \) and \( b = \pi \) and so

\[
f(t + \pi) = \sin[0.5(t + \pi)] = \sin\left(\frac{t}{2} + \frac{\pi}{2}\right) = \cos\frac{t}{2}.
\]

Hence,

\[
\mathcal{L}\{\sin(0.5t) \cdot u(t - \pi)\} = e^{-\pi s}\mathcal{L}\left\{\cos\frac{t}{2}\right\} = e^{-\pi s} \cdot \frac{s}{s^2 + \frac{1}{4}}.
\]
Example 10.27. Find the solution of

\[ \dot{x} + 9x = 2\delta(t - 3) \]

satisfying the initial condition \( x(0) = -5 \).

Solution. Taking the Laplace transform and using the linearity property as well as the transform of the first derivative property, we obtain

\[ sX(s) - x(0) + 9X(s) = 2e^{-3s}, \]

where as usual we use the alternate notation \( X(s) \) in lieu of \( \mathcal{L}\{x(t)\} \). As \( x(0) = -5 \), we have

\[ (s + 9)X(s) + 5 = 2e^{-3s} \]

and so

\[ X(s) = -\frac{5}{s + 9} + 2e^{-3s} \]

(10.6.5)

The second term on the right-hand side of (10.6.5) involves \( e^{-bs}\mathcal{L}\{f(t)\} \) where \( b = 3 \) and \( \mathcal{L}\{f(t)\} = 1/(s + 9) \). Hence, \( f(t) = e^{-9t} \). According to shift property (10.6.4),

\[ e^{-bs}\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t - b)u(t - b)\}, \]

Since \( f(t - 3) = e^{-9(t-3)} \),

\[ \frac{e^{-3s}}{s + 9} = e^{-3s}\mathcal{L}\{e^{-9t}\} = \mathcal{L}\{e^{-9(t-3)}u(t-3)\}. \]

It follows that (10.6.5) can be written as

\[ \mathcal{L}\{x(t)\} = \mathcal{L}\{-5e^{-9t} + 2e^{-9(t-3)}u(t-3)\}. \]

Therefore, the solution of the initial value problem is

\[ x(t) = -5e^{-9t} + 2e^{-9(t-3)}u(t-3) = \begin{cases} -5e^{-9t}, & \text{if } t < 3 \\ -5e^{-9t} + 2e^{-9(t-3)}, & \text{if } t > 3. \end{cases} \]

10.6.3 Derivatives of Transforms

Is the derivative of the Laplace transform of a function of any use? Let’s look into this. Formally differentiating the Laplace transform \( F(s) \) of a function \( f(t) \) with respect to \( s \), we obtain

\[ F'(s) = \frac{d}{ds}F(s) = \frac{d}{ds}\mathcal{L}\{f(t)\}(s) = \frac{d}{ds}\int_0^{\infty} e^{-st}f(t)\,dt \]

(10.6.6)

\[ = \int_0^{\infty} \frac{\partial}{\partial s}[e^{-st}f(t)]\,dt = \int_0^{\infty} e^{-st}(-t)f(t)\,dt = \mathcal{L}\{-tf(t)\}. \]
The step where we move \( \frac{d}{ds} \) after the integral sign and change it to \( \frac{\partial}{\partial s} \) reverses the order of differentiation and integration. This seems reasonable but that in itself does not make it true. Although we are not in a position to prove it because of the advanced mathematics that would be needed, this step is valid for the integrand \( e^{-st} f(t) \) if the function \( f(t) \) is piecewise continuous and of exponential order. Consequently, if that is the case, we conclude from (10.6.6) that

\[
\mathcal{L}\{tf(t)\} = F'(s).
\] (10.6.7)

Are there similar formulas when higher powers of \( t \) are involved? To answer that, consider \( \mathcal{L}\{t^n f(t)\} \). By (10.6.7),

\[
\mathcal{L}\{(t)^2 f(t)\} = \mathcal{L}\{(-t)[tf(t)]\} = \frac{d}{ds}\mathcal{L}\{tf(t)\} = \frac{d}{ds}[F'(s)] = F''(s).
\]

Similarly, it can be shown that \( \mathcal{L}\{(t)^3 f(t)\} = F'''(s) \). The pattern that is developing suggests the formula

\[
\mathcal{L}\{(t)^n f(t)\} = F^{(n)}(s)
\] (10.6.8)

when \( n \) is a positive integer. We now prove this is the case by mathematical induction.

Suppose (10.6.8) is true for an integer \( n = k \) (as it is for \( n = 1, 2, 3 \)). Then for \( n = k + 1 \), it follows from (10.6.7) and (10.6.8) that

\[
\mathcal{L}\{(t)^{k+1} f(t)\} = \mathcal{L}\{(t)[(t)^{k} f(t)]\} = \frac{d}{ds}\mathcal{L}\{(t)^{k} f(t)\} = \frac{d}{ds}F^{(k)}(s) = F^{(k+1)}(s).
\]

So if (10.6.8) is true for \( n = k \), logic demands that it also be true for \( n = k + 1 \). Therefore, since it has already been established that (10.6.8) is true for \( n = 1, 2, 3 \), it follows that it also must be true for the rest of the positive integers.

In short, because of the validity of differentiating inside the integral sign and the induction argument, we have the following property.

**Derivatives of Transforms**

**Theorem 10.8.** Suppose a function \( f(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \gamma \). Let \( F(s) = \mathcal{L}\{f(t)\} \). Then

\[
\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)
\] (10.6.9)

for \( s > \gamma \) and \( n = 1, 2, 3, \ldots \).

**Example 10.28.** Find \( \mathcal{L}\{t \sin bt\} \).

\[\text{15} \text{Churchill [16, Sec. 12] states and proves such a result.}\]
10.6. PROPERTIES OF THE LAPLACE TRANSFORM

Solution. Clearly Theorem 10.8 applies here since \( \sin bt \) is multiplied by \( t \) and we know that \( \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2} \). In the notation of the theorem, \( f(t) = \sin bt \) and \( F(s) = \frac{b}{s^2 + b^2} \). Setting \( n = 1 \) in (10.6.9), we have

\[
\mathcal{L}\{t \sin bt\} = \frac{d}{ds} F(s) = -\frac{2bs}{(s^2 + b^2)^2}.
\]

Therefore,

\[
\mathcal{L}\{t \sin bt\} = \frac{2bs}{(s^2 + b^2)^2}.
\]

Example 10.29. Find \( \mathcal{L}\{t^2 e^{3t}\} \).

Solution. Since \( \mathcal{L}\{e^{3t}\} = 1/(s-3) \),

\[
\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{-t e^{3t}\} = \frac{d^2}{ds^2} \left( \frac{1}{s-3} \right) = \frac{d}{ds} \left( \frac{-1}{(s-3)^2} \right) = \frac{2}{(s-3)^3}.
\]

Another approach is to use the shift property in the s-domain; it is also less work. Since \( \mathcal{L}\{t^2\} = 2/s^3 \), it immediately follows from shifting \( s \) that

\[
\mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}.
\]

10.6.4 Integral of a Transform

We just saw that multiplying a function \( f \) by \( t \) corresponds to the negative of the derivative of the transform of \( f \). Now we will see that dividing \( f \) by \( t \) corresponds to an integral of its transform—precisely what integral is the subject of the next theorem.

**Integral of a Transform**

**Theorem 10.9.** Suppose a function \( f(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \gamma \). Furthermore, suppose

\[
\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to 0^+} \frac{F(s)}{s} \quad (10.6.10)
\]

exists and is finite. Then

\[
\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \frac{1}{s} \int_{s}^{\infty} F(u) \ du, \quad (10.6.11)
\]

where \( F(s) = \mathcal{L}\{f\}(s) \).
Proof. Integrating \( F(u) = \int_0^\infty e^{-ut} f(t) \, dt \) from \( u = s \) to \( u = T \) where \( T > s \), we obtain
\[
\int_s^T F(u) \, du = \int_s^T \left( \int_0^\infty e^{-ut} f(t) \, dt \right) \, du.
\]
The hypotheses of Theorem 10.9 allows the order of integration to be reversed: \( \int_s^T \frac{F(u)}{u} \, du = \int_0^T f(t) \left( \int_s^T e^{-ut} \, du \right) \, dt. \) \hspace{1cm} (10.6.12)

The piecewise continuity of \( f(t)/t \) on \( [0, \infty) \) and (10.6.10) implies that \( f(t)/t \) is piecewise continuous on \( [0, \infty) \). Since \( f(t) \) is of exponential order, it is easy to show that \( f(t)/t \) is of exponential order. Thus, the Laplace transform of \( f(t)/t \) exists. Because of this and as
\[
\int_s^T e^{-ut} \, du = -\frac{1}{t} e^{-ut}\bigg|_s^T = \frac{1}{t} e^{-st} - \frac{1}{t} e^{-Tt},
\]
we can write (10.6.12) as
\[
\int_s^T F(u) \, du = \int_0^\infty f(t) \left( \int_s^T e^{-ut} \, du \right) \, dt - \int_0^\infty f(t) \, dt
\]
\[= \mathcal{L} \left\{ \frac{f(t)}{t} \right\} (s) - \mathcal{L} \left\{ \frac{f(t)}{t} \right\} (T).\]
This implies (10.6.11) since \( \mathcal{L} \{ f(t)/t \}(T) \to 0 \) as \( T \to \infty \) by (10.5.3).

Example 10.30. Find the Laplace transform of \( \frac{e^t - 1}{t} \).

Solution. In the notation of Theorem 10.9,
\[ f(t) = e^t - 1 \]
and
\[ F(s) = \mathcal{L} \{ f(t) \} = \frac{1}{s - 1} - \frac{1}{s}. \]
By L'Hôpital's Rule,
\[ \lim_{t \to 0^+} \frac{e^t - 1}{t} = \lim_{t \to 0^+} e^t = 1. \]
Consequently, the hypotheses of Theorem 10.9 are satisfied. Integrating \( F(s) \), we obtain
\[
\int_s^T F(u) \, du = \int_s^T \left( \frac{1}{u - 1} - \frac{1}{u} \right) \, du = \ln \left( \frac{u - 1}{u} \right) \bigg|_s^T
\]
\[= \ln \left( \frac{T - 1}{T} \right) - \ln \left( \frac{s - 1}{s} \right).\]

\(16\) The reason for this is found in Churchill [16, Sec. 18]. However, it may be difficult for a reader at this level to understand since uniform convergence is used, a concept that is introduced in advanced calculus or real analysis textbooks, such as Bartle and Sherbert [5], Fulks [25], and Ross [46].
Thus,
\[ \int_s^\infty F(u) \, du = \lim_{T \to \infty} \int_s^T F(u) \, du = -\ln \left( \frac{s-1}{s} \right) = \ln \left( \frac{s}{s-1} \right) \]
as
\[ \lim_{T \to \infty} \ln \left( \frac{T-1}{T} \right) = \lim_{T \to \infty} \ln \left( 1 - \frac{1}{T} \right) = \ln 1 = 0. \]

Therefore,
\[ \mathcal{L} \left\{ \frac{e^t - 1}{t} \right\} = \ln \left( \frac{s}{s-1} \right). \]

**Example 10.31.** Find
\[ \mathcal{L} \left\{ \frac{\sin bt}{t} \right\}, \]
where \( b \) is a constant.

**Solution.** The hypotheses of Theorem 10.9 are satisfied since \( \sin bt \) is continuous, \( \sin bt = O(1) \), and (10.6.10) holds:
\[ \lim_{t \to 0^+} \frac{\sin bt}{t} = \lim_{t \to 0^+} \left( b - \frac{b^3 t^2}{3!} + \frac{b^5 t^4}{5!} - \frac{b^7 t^6}{7!} + \cdots \right) = b. \]

By (10.6.11),
\[ \mathcal{L} \left\{ \frac{\sin bt}{t} \right\} = \int_s^\infty \frac{b}{u^2 + b^2} \, du = \lim_{T \to \infty} \left( \tan^{-1} \frac{T}{b} - \tan^{-1} \frac{s}{b} \right) = \frac{\pi}{2} - \tan^{-1} \frac{s}{b}. \]

As \( \tan^{-1} x + \tan^{-1} (x^{-1}) = \pi/2 \),
\[ \mathcal{L} \left\{ \frac{\sin bt}{t} \right\} = \tan^{-1} \frac{b}{s}. \tag{10.6.13} \]

### 10.6.5 Transform of an Integral

Assuming a function \( f(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \gamma \), another useful property can be derived by applying the Transform of the First Derivative formula (10.4.7) to the integral function \( g(t) = \int_0^t f(\tau) \, d\tau \). Since \( f \) is integrable on \([0, \infty)\), \( g \) is continuous on \([0, \infty)\); moreover, by one of the versions of the Fundamental Theorem of Calculus, \( g \) is differentiable at those places where \( f \) is continuous, in which case\(^{17}\)

\[ g'(t) = \frac{d}{dt} \int_0^t f(\tau) \, d\tau = f(t). \]

---
\(^{17}\)See Bartle and Sherbert [5, pp. 257-258], Ross [46, p. 200], or Gaughan [26, p. 160].
CHAPTER 10. THE LAPLACE TRANSFORM

From this it follows that \( g' \) is piecewise continuous on \([0, \infty)\).

Is \( g \) of exponential order? If it is, then this along with its piecewise smoothness will guarantee the existence of its transform. Since \( f \) is of exponential order \( \gamma \), there are constants \( M_1 > 0 \) and \( t_1 \geq 0 \) such that \( |f(\tau)| \leq M_1 e^{\gamma \tau} \) for \( \tau \geq t_1 \). If the constant \( \gamma \) is not already positive, we can certainly replace it with a positive number. Since \( f \) is piecewise continuous, it is bounded on the finite interval \([0, t_1]\). In other words, there is a constant \( M_2 \) such that \( |f(\tau)| \leq M_2 e^{\gamma \tau} \) for \( 0 \leq \tau \leq t_1 \). Even more so then, \(|f(\tau)| \leq M_2 e^{\gamma \tau} \) on \([0, \infty)\). Since \( -|f(\tau)| \leq f(\tau) \leq |f(\tau)| \),

\[
\left| \int_0^t f(\tau) \, d\tau \right| \leq \int_0^t |f(\tau)| \, d\tau.
\]

Thus,

\[
|g(t)| \leq \int_0^t |f(\tau)| \, d\tau \leq M \int_0^t e^{\gamma \tau} \, d\tau \leq \frac{M}{\gamma} e^{\gamma t}
\]

for all \( t \geq 0 \). Hence, \( g \) is of exponential order \( \gamma \) for \( s > \gamma \).

Generally there are points in \([0, \infty)\) where \( g' \) is not differentiable; that depends on \( f \) and the location of its discontinuities. However, as we concluded above, \( g' \) is piecewise continuous on \([0, \infty)\). This, that \( g(t) = O(e^{\gamma t}) \), and the continuity of \( g \) on \([0, \infty)\) guarantee that formula (10.4.7) still applies.\(^{18}\) Therefore,

\[
\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)
\]

for \( s > \gamma \). Since \( g(0) = 0 \),

\[
\mathcal{L}\{f(t)\} = s\mathcal{L}\{g(t)\} = s\mathcal{L}\left\{ \int_0^t f(\tau) \, d\tau \right\}.
\]

As a result, we have the following property.

**Transform of an Integral**

**Theorem 10.10.** If \( f(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \gamma \), then

\[
\mathcal{L}\left\{ \int_0^t f(\tau) \, d\tau \right\} = \frac{1}{s} \mathcal{L}\{f(t)\}
\]

for \( s > \gamma \).

**Example 10.32.** For \( f(t) \) in (10.6.14), let \( f(t) \equiv 1 \) on \([0, \infty)\). It clearly satisfies the hypotheses of Theorem 10.10. Hence,

\[
\mathcal{L}\left\{ \int_0^t 1 \, d\tau \right\} = \frac{1}{s} \mathcal{L}\{1\}.
\]

\(^{18}\)See Churchill [16, p. 8].
Since \( \int_0^t d\tau = t \) and \( L\{1\} = 1/s \), we conclude
\[
L\{t\} = \frac{1}{s^2} \quad \text{for} \quad s > 0.
\]

Even though we obtained this result many pages ago in Section 10.2, we chose it as our first example in order to become comfortable with Theorem 10.10 with a familiar result. The next example is new and not as easy.

**Example 10.33.** Find the Laplace transform of the sine-integral function
\[
\text{Si}(t) = \int_0^t \frac{\sin \tau}{\tau} \, d\tau.
\]

**Solution.** The integrand \( t^{-1} \sin t \) is piecewise continuous on \([0, \infty)\)
\[
\lim_{t \to 0^+} \frac{\sin t}{t} = 1.
\]

And it is of exponential order \( \gamma = 0 \) because
\[
\left| \frac{\sin t}{t} \right| \leq \frac{1}{t} \leq 1 \quad \text{for} \quad t \geq 1.
\]

Consequently, the conditions of Theorem 10.10 are satisfied, By (10.6.10),
\[
L \{ \text{Si}(t) \} = L \left\{ \int_0^t \frac{\sin \tau}{\tau} \, d\tau \right\} = \frac{1}{s} L \left\{ \frac{\sin t}{t} \right\} = \frac{1}{s} \tan^{-1} \frac{1}{s}
\]
for \( s > 0 \), where we have used (10.6.13) in the last step.

### 10.6.6 Transform of a Periodic Function

In this section, we will learn how to find Laplace transforms of piecewise continuous functions that are also periodic. A function \( f \) is said to be **periodic** on \([0, \infty)\) if a constant \( b \) exists such that
\[
f(t + b) = f(t)
\]
at all values of \( t \in [0, \infty) \) for which \( f(t) \) is defined. The smallest positive value of \( b \) for which (10.6.15) holds is called the **period** of \( f \). For example, the cosine and sine functions are periodic with period \( 2\pi \).

For a given periodic function \( f \) with period \( b \), let \( f_E \) be the function that extends the rule for \( f \) on the interval \([0, b)\) to the entire interval \([0, \infty)\).

**Example 10.34.** Let \( f \) be the periodic function with period 1, where the rule for \( f \) on \([0, 1)\) is
\[
f(t) = t.
\]
Then the function \( f_E \) is
\[
f_E(t) = t
\]
for \( 0 \leq t < \infty \). The graphs of \( f \) and \( f_E \) are shown side by side in Figure 10.1.
Theorem 10.11. If \( f(t) \) is periodic with period \( b \) and piecewise continuous on \( [0, b) \), then

\[
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-bs}} \int_0^b e^{-st} f(t) \, dt
\]

(10.6.16)

\[
= \frac{1}{1 - e^{-bs}} \left[ \mathcal{L}\{f_E(t)\} - e^{-bs} \mathcal{L}\{f_E(t + b)\} \right],
\]

(10.6.17)

where \( f_E \) is the function that extends the rule for \( f \) on \( [0, b) \) to \([0, \infty)\).

Proof. Piecewise continuity of \( f \) on \([0, b)\) and its periodicity of \( b \) imply \( f \) is bounded and piecewise continuous on all of \([0, \infty)\). Therefore, by Corollary 10.2, the transform of \( f \) exists, where

\[
\mathcal{L}\{f(t)\} = \int_0^b e^{-st} f(t) \, dt + \int_b^\infty e^{-st} f(t) \, dt.
\]

With the change of variable \( u = t - b \), the improper integral becomes

\[
\int_b^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-s(u+b)} f(u + b) \, du = e^{-sb} \int_0^b e^{-su} f(u) \, du
\]

as the period of \( f \) is \( b \). Therefore,

\[
\mathcal{L}\{f(t)\} = \int_0^b e^{-st} f(t) \, dt + e^{-sb} \mathcal{L}\{f(t)\},
\]
from which (10.6.16) follows. To obtain (10.6.17), first observe that
\[ \int_0^b e^{-st} f(t) \, dt = \int_0^b e^{-st} f_E(t) \, dt = \int_0^\infty e^{-st} f_E(t)[1 - u(t - b)] \, dt \]
as \( f_E(t) \equiv f(t) \) on \([0, b]\) and
\[ f_E(t)[1 - u(t - b)] = \begin{cases} f_E(t), & \text{if } 0 \leq t < b \\ 0, & \text{if } t > b. \end{cases} \]

Thus,
\[ \int_0^b e^{-st} f(t) \, dt = \int_0^\infty e^{-st} f_E(t) \, dt - \int_0^\infty e^{-st} f_E(t) u(t - b) \, dt \]
\[ = \mathcal{L}\{f_E(t)\} - \mathcal{L}\{f_E(t)u(t - b)\} = \mathcal{L}\{f_E(t)\} - e^{-bs} \mathcal{L}\{f_E(t + b)\} \]
by the shift in the \( t \)-domain property.

**Example 10.35.** Find the Laplace transform of the square-wave function whose graph is shown in Figure 10.2.

![Fig. 10.2: A periodic square wave](image)

**Solution.** In order to apply Theorem 10.11, we must have a formula for the function. Let’s call it \( f \). It is apparent from the graph that \( f \) is periodic with period 2 and
\[ f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \]
for $0 < t < 2$. By (10.6.16) with $b = 2$:

$$
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} f(t) \, dt
$$

$$
= \frac{1}{1 - e^{-2s}} \left( \int_0^1 e^{-st} \cdot 1 \, dt + \int_1^2 e^{-st} \cdot 0 \, dt \right)
$$

$$
= \frac{1}{1 - e^{-2s}} \left( \frac{1}{s} - \frac{1}{s} e^{-s} \right) = \frac{1 - e^{-s}}{s(1 - e^{-2s})}.
$$

**Example 10.36.** Find the Laplace transform of the function $f$ that is periodic with period 1 and with the values $f(t) = t$ for $0 < t < 1$.

**Solution 1.** We have already seen the graph of $f$ in Figure 10.1. By (10.6.16) with $b = 1$,

$$
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-s}} \int_0^1 e^{-st} t \, dt.
$$

Integration by parts yields

$$
\int_0^1 e^{-st} t \, dt = \frac{1}{s^2} (1 - e^{-s}) - \frac{1}{s} e^{-s}.
$$

Thus,

$$
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-s}} \left[ \frac{1 - e^{-s}}{s^2} - \frac{1}{s} e^{-s} \right] = \frac{1}{s^2} - \frac{1}{s(e^s - 1)}.
$$

**Solution 2.** Alternatively, we can use (10.6.17) and avoid integrating by parts. Since $b = 1$ and $f_E(t) = t$ for $0 < t < \infty$, $f_E(t + b) = f_E(t + 1) = t + 1$.

Thus,

$$
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-s}} [\mathcal{L}\{f(t)\} - e^{-s} \mathcal{L}\{f(t + 1)\}] = \frac{1}{1 - e^{-s}} \left[ \frac{1}{s^2} - e^{-s} \left( \frac{1}{s^2} - \frac{1}{s} \right) \right]
$$

$$
= \frac{1}{s^2} - \frac{1}{s(e^s - 1)}.
$$

**Example 10.37.** Find $\mathcal{L}\{g(t)\}$, where $g(t)$ is periodic with period $\pi$ and

$$
g(t) = \sin t
$$

for $0 < t < \pi$. Its graph is the full-rectified sine wave shown in Figure 10.3.
10.6. PROPERTIES OF THE LAPLACE TRANSFORM

Fig. 10.3: A full-rectified sine wave

Solution. Applying (10.6.17) with \( b = \pi \) and \( g_E(t) = \sin t \), we have

\[
\mathcal{L}\{g(t)\} = \frac{1}{1 - e^{-\pi s}} \left[ \mathcal{L}\{g_E(t)\} - e^{-\pi s} \mathcal{L}\{g_E(t + \pi)\} \right]
\]

or

\[
\mathcal{L}\{g(t)\} = \frac{1}{1 - e^{-\pi s}} \left[ \mathcal{L}\{\sin(t)\} - e^{-\pi s} \mathcal{L}\{\sin(t + \pi)\} \right].
\]

As \( \sin(t + \pi) = -\cos t \),

\[
\mathcal{L}\{g(t)\} = \frac{1}{1 - e^{-\pi s}} \left[ \frac{1}{s^2 + 1} + e^{-\pi s} \frac{s}{s^2 + 1} \right] = \frac{1 + e^{-\pi s}}{(s^2 + 1)(1 - e^{-\pi s})}.
\]

10.6.7 Convolution

Consider the initial value problem

\[
\frac{dx}{dt} - x = g(t), \quad x(0) = 0,
\]

where \( g \) is piecewise continuous and of exponential order on \([0, \infty)\). Taking the transform of both sides of the equation in (10.6.18), using the initial condition, and solving for \( \mathcal{L}\{x(t)\} \), we get

\[
\mathcal{L}\{x(t)\} = \frac{1}{s - 1} \mathcal{L}\{g(t)\}
\]

or

\[
\mathcal{L}\{x(t)\} = \mathcal{L}\{e^t\} \mathcal{L}\{g(t)\}
\]

\(^{19}\)A full-rectified sine wave is obtained from a sine wave by inverting all of its negative (or positive) half-cycles.
as \(1/(s-1) = \mathcal{L}\{e^t\}\). This says the transform of the solution \(x(t)\) of (10.6.18) is the product of the transform of \(e^t\) and that of \(g(t)\). At this point we do not yet know what to do next to find \(x(t)\) itself even if \(g(t)\) were known. So let’s leave this for a moment and solve (10.6.18) by the integrating factor method or with the variation of parameters formula given in Chapter 5. Using either method, we find

\[
x(t) = \int_0^t e^{t-u}g(u) \, du. \tag{10.6.20}
\]

Comparing the Laplace transform of (10.6.19) to (10.6.20), we see

\[
\mathcal{L}\{e^t\} \mathcal{L}\{g(t)\} = \mathcal{L}\left\{ \int_0^t e^{t-u}g(u) \, du \right\}. \tag{10.6.21}
\]

The point to be made is that the solutions of some initial value problems, such as (10.6.18), can be expressed by integrals of the form

\[
\int_0^t f(t-u)g(u) \, du.
\]

In this example, \(f(t) = e^t\). Integrals of this type are known as convolution integrals as we formally define next.

### Convolution Integrals

**Definition 10.4.** Let \(f(t)\) and \(g(t)\) be piecewise continuous functions on \([0, \infty)\). The convolution of \(f\) and \(g\) is the integral function \(h\) defined by

\[
h(t) := \int_0^t f(t-u)g(u) \, du. \tag{10.6.22}
\]

**Example 10.38.** Find the convolution of \(f(t) = t\) and \(g(t) = e^t\). 

**Solution.** By (10.6.22) and integration by parts, we obtain

\[
h(t) = \int_0^t (t-u)e^u \, du = [(t-u)e^u + e^u]_0^t = e^t - t - 1.
\]

Instead of \(h\) or some other letter to denote the convolution integral, it is standard practice to use the symbol \(f \ast g\):

\[
(f \ast g)(t) := \int_0^t f(t-u)g(u) \, du.
\]

With this notation, the convolution of \(t\) and \(e^t\) is written as follows:

\[
t \ast e^t = \int_0^t (t-u)e^u \, du = e^t - t - 1.
\]

Convolution has the properties of commutativity, associativity, and distributivity as set forth in the next theorem.
10.6. PROPERTIES OF THE LAPLACE TRANSFORM

Properties of Convolution

**Theorem 10.12.** If \( f(t), g(t), \) and \( h(t) \) are piecewise continuous functions on \([0, \infty)\), then the convolution operator has the following properties:

(i) **Commutativity:** \( f * g = g * f \)

(ii) **Associativity:** \( f * (g * h) = (f * g) * h \)

(iii) **Distributivity:** \( f * (g + h) = f * g + g * h \)

(iv) \( 0 * f = f * 0 = 0 \)

**Proof.** To prove (i), simply change variables: \( v = t - u \). Then

\[
(f \ast g)(t) = \int_0^t f(t-u)g(u)\,du = -\int_0^t f(v)g(t-v)\,dv = \int_0^t g(t-v)f(v)\,dv = (g \ast f)(t).
\]

The proof of (iv) is trivial:

\[
(f \ast 0)(t) = \int_0^t f(t-u) \cdot 0\,du = \int_0^t 0\,du = 0,
\]

which together with (i) proves (iv). The proofs of (ii) and (iii) are left as exercises.

Even though it appears from Theorem 10.12 that the convolution operator \( \ast \) has all of the properties of ordinary multiplication, it does not! For example, \( 1 \ast 1 = 1 \) whereas \( 1 \neq 1 \) as

\[
1 \ast 1 = \int_0^t 1 \cdot 1\,du = t.
\]

For the exponential function \( f(t) = e^t \), we determined from (10.6.21) that

\[
\mathcal{L}\{ (f \ast g)(t) \} = \mathcal{L}\{ f(t) \} \mathcal{L}\{ g(t) \}.
\]

In fact, as we prove next, this is true for any two piecewise continuous functions of exponential order.

**Convolution Theorem**

**Theorem 10.13.** Let \( f(t) \) and \( g(t) \) be piecewise continuous functions on \([0, \infty)\) and of exponential order \( \gamma \). Then

\[
\mathcal{L}\{ (f \ast g)(t) \} = \mathcal{L}\{ f(t) \} \mathcal{L}\{ g(t) \} \tag{10.6.23}
\]

for \( s > \gamma \).
Proof. By Corollary 10.2, the Laplace transforms
\[ \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt =: F(s) \]
and
\[ \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t) \, dt =: G(s) \]
exist for \( s > \gamma \). Starting with the left-hand side of (10.6.23), we have
\[
\mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = F(s) \int_0^\infty e^{-st} g(t) \, dt = \int_0^\infty e^{-st} F(s) g(t) \, dt = \int_0^\infty e^{-su} F(s) g(u) \, du.
\]
Note that we can move \( F(s) \) inside the above integral because it does not depend on the variable of integration. Consequently,
\[
\mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = \int_0^\infty e^{-su} \left( \int_0^\infty e^{-st} f(t) \, dt \right) g(u) \, du = \int_0^\infty e^{-su} g(u) \, du.
\]
With the change of variable \( v = t + u \), this becomes
\[
\mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = \int_0^\infty \left( \int_u^\infty e^{-sv} f(v - u) \, dv \right) g(u) \, du. \tag{10.6.24}
\]
The region of integration is shown in Figure 10.4. It is the shaded \( v \)-shaped region that extends infinitely to the right.

![Fig. 10.4: Region of integration for (10.6.24)](image)

This is where the proof gets technical and goes beyond the mathematical level of this book. Suffice it to say that it can be argued that it is legitimate to change the order of integration in (10.6.24) (cf. [16, Sec. 13]). Thus,
10.6. PROPERTIES OF THE LAPLACE TRANSFORM

\[
\mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = \int_0^\infty e^{-sv} \left( \int_0^v f(v-u)g(u) \, du \right) \, dv \\
= \int_0^\infty e^{-sv}(f \ast g)(v) \, dv = \int_0^\infty e^{-st}(f \ast g)(t) \, dt \\
= \mathcal{L}\{(f \ast g)(t)\}
\]

for \( s > \gamma \).

One important application of the convolution theorem (10.6.23) is that it can be used to solve certain types of integral equations. An integral equation is an equation with the unknown function appearing under an integral sign. One such integral equation has the form

\[
x(t) = f(t) + \int_0^t k(t-u)x(u) \, du,
\]

(10.6.25)

where \( f \) and \( k \) are known functions. The function \( k \) is called the kernel. This particular type of equation is referred to as a Volterra integral equation of convolution type. The phrase “of convolution type” means that the kernel \( k \) depends only on the combination \( t - u \). In the following example, we illustrate how to solve such an equation using the convolution theorem.

**Example 10.39.** Solve the Volterra integral equation

\[
x(t) = t + \int_0^t (t-u)x(u) \, du.
\]

(10.6.26)

**Solution.** First take the Laplace transform of (10.6.26) and apply (10.6.23):

\[
\mathcal{L}\{x(t)\} = \mathcal{L}\{t\} + \mathcal{L} \left\{ \int_0^t (t-u)x(u) \, du \right\}
\]

\[
= \mathcal{L}\{t\} + \mathcal{L}\{t\} \mathcal{L}\{x(t)\} = \frac{1}{s^2} + \frac{1}{s^2} \mathcal{L}\{x(t)\}.
\]

Now solve for \( \mathcal{L}\{x(t)\} \):

\[
\mathcal{L}\{x(t)\} \left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2} \Rightarrow \mathcal{L}\{x(t)\} = \frac{1}{s^2 - 1}.
\]

From the partial fraction expansion

\[
\frac{1}{s^2 - 1} = \frac{1}{2} \left( \frac{1}{s-1} \right) - \frac{1}{2} \left( \frac{1}{s+1} \right),
\]

we see that

\[
\frac{1}{s^2 - 1} = \frac{1}{2} \mathcal{L}\{e^t\} - \frac{1}{2} \mathcal{L}\{e^{-t}\}.
\]
Therefore, \[
\mathcal{L}\{x(t)\} = \mathcal{L}\left\{ \frac{1}{2} e^t - \frac{1}{2} e^{-t} \right\}
\]
from which we conclude the solution is
\[
x(t) = \frac{e^t - e^{-t}}{2} = \sinh t.
\]

There are also some types of integro-differential equations that can be solved with Laplace transforms. **Integro-differential equations** involves both derivatives and integrals of the unknown function. One type that we can solve with transforms is a **Volterra integro-differential equation of convolution type** of the form
\[
x'(t) = f(t) + \int_0^t k(t-u)x(u)\,du.
\]

**Example 10.40.** Find the solution of the Volterra integro-differential equation
\[
x'(t) = t - \int_0^t e^{t-u}x(u)\,du
\]
satisfying the initial condition \(x(0) = 0\).

*Solution.* The Laplace transform of the integro-differential equation is
\[
\mathcal{L}\{x'(t)\} = \mathcal{L}\{t\} - \mathcal{L}\left\{ \int_0^t e^{t-u}x(u)\,du \right\}
\]
or
\[
s\mathcal{L}\{x(t)\} - x(0) = \frac{1}{s^2} - \mathcal{L}\{e^t\} \mathcal{L}\{x(t)\}
\]
by the convolution theorem. Using the initial condition \(x(0) = 0\) and writing this in terms of \(X(s) := \mathcal{L}\{x(t)\}\), we have
\[
sX(s) = \frac{1}{s^2} - \frac{1}{s-1} X(s).
\]
Hence,
\[
X(s) = \frac{s-1}{s^2(s^2-s+1)}.
\]
A partial fraction expansion of \(X(s)\) is
\[
X(s) = -\frac{1}{s^2} + \frac{1}{s^2-s+1}.
\]
In Example 10.24, we determined that
\[
\frac{1}{s^2-s+1} = \mathcal{L}\left\{ \frac{2}{\sqrt{3}} e^{\frac{4}{3}t} \sin \left( \frac{\sqrt{2}}{2}t \right) \right\}.
\]
Therefore, the solution is
\[
x(t) = -t + \frac{2}{\sqrt{3}} e^{\frac{4}{3}t} \sin \left( \frac{\sqrt{2}}{2}t \right).
\]
10.7 Inverse Laplace Transforms

When solving a first-order initial value problem with Laplace transforms, we eventually arrive at the point where we have determined the Laplace transform of the solution. For instance, in Example 10.18, we considered the initial value problem

\[ x' + 4x = 2 - 25 \cos 3t, \quad x(0) = 2. \]

The transform of the solution, denoted by \( X(s) \), turned out to be

\[ X(s) = \frac{2}{s + 4} + \frac{2}{s(s + 4)} - \frac{25s}{(s + 4)(s^2 + 9)}. \]

By Lerch’s theorem, there is only one continuous function \( x(t) \) whose Laplace transform is equal to \( X(s) \). This function \( x(t) \) is known as the inverse Laplace transform of the function \( X(s) \) and is the solution of the initial value problem. Even though we had not used this term before, the last step in solving a differential equation by the Method of Laplace Transforms is to determine an inverse Laplace transform.

**Inverse Laplace Transform**

**Definition 10.5.** An inverse Laplace transform of a function \( X(s) \) is a function \( x(t) \), if it exists, whose Laplace transform is \( X(s) \). If \( \mathcal{L}\{x(t)\} = X(s) \), then we write

\[ \mathcal{L}^{-1}\{X(s)\} = x(t). \]

**Example 10.41.** Find \( \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 9} \right\} \).

**Solution.** Since \( \mathcal{L}\{ \cos 3t \} = s/(s^2 + 3^2) \),

\[ \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 9} \right\} = \cos 3t. \]

**Example 10.42.** Find \( \mathcal{L}^{-1}\left\{ \frac{6}{s^4} \right\} \).

**Solution.** Recall that \( \mathcal{L}\{t^n\} = n!/s^{n+1} \). In particular, \( \mathcal{L}\{t^3\} = 3!/s^4 \). Thus,

\[ \mathcal{L}^{-1}\left\{ \frac{6}{s^4} \right\} = \mathcal{L}^{-1}\left\{ \frac{3!}{s^4} \right\} = t^3. \]

Note that functions do not have unique inverse transforms. For example, the Laplace transform of the continuous function \( t^3 \) and of the piecewise continuous function

\[ h(t) := \begin{cases} t^3, & \text{if } t \neq 2, 7 \\ -1, & \text{if } t = 2 \\ 10, & \text{if } t = 7 \end{cases} \]
is \(3!/s^4\). So do we say
\[
\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3 \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = h(t)?
\]

Moreover, there are infinitely many other possibilities too. Which of them is the inverse Laplace transform? The answer is found in our objective of solving linear differential equations. When the function \(g(t)\) is continuous, solutions of linear first-order initial value problems of the form
\[
a \frac{dx}{dt} + bx = g(t), \quad x(t_0) = x_0
\]
or of linear second-order initial value problems of the form
\[
a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = g(t), \quad x(t_0) = x_0, \quad x'(t_0) = x_1
\]
are continuous functions. Obtaining these solutions by the method of Laplace transforms necessitates choosing the inverse transform that is a continuous function. Lerch’s theorem implies that a function has a unique continuous inverse transform, if it has one at all. For that reason, we stipulate the following convention when solving differential equations with Laplace transforms.

### Convention for Choosing Inverse Laplace Transforms

- The **inverse Laplace transform** of a function \(X(s)\) is the unique function \(x(t)\) that is continuous on the interval \([0, \infty)\) whose Laplace transform is \(X(s)\); that is,
\[
\mathcal{L}\{x(t)\} = X(s),
\]
providing such a continuous function \(x(t)\) exists.
- If every function satisfying (10.7.1) is discontinuous on \([0, \infty)\), the inverse Laplace transform of \(X(s)\) is any piecewise continuous function \(x(t)\) satisfying (10.7.1).

**Example 10.43.** Find \(\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}\).

**Solution.** Since \(\mathcal{L}\{u(t - b)\} = e^{-bs}/s\), it follows that
\[
\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} = u(t - 3) = \begin{cases} 0, & \text{if } t < 3 \\ 1, & \text{if } t \geq 3. \end{cases}
\]

The shift properties are indispensable, especially when it comes to finding inverse transforms, as illustrated by the following examples.
Example 10.44. Find $\mathcal{L}^{-1}\{X(s)\}$ for $X(s) = \frac{s - 3}{(s - 3)^2 + 25}$.

Solution. Except for the shift of 3 units in $s$, $X(s)$ is $s/(s^2 + 25)$, which is the Laplace transform of $\cos 5t$:

$$\frac{s}{s^2 + 5^2} = \mathcal{L}\{\cos 5t\}.$$ 

It follows from (10.6.1), the shift property in $s$, that

$$\frac{s - 3}{(s - 3)^2 + 5^2} = \mathcal{L}\{e^{3t}\cos 5t\}$$

and so

$$\mathcal{L}^{-1}\{X(s)\} = e^{3t}\cos 5t.$$

Example 10.45. Find $\mathcal{L}^{-1}\{Y(s)\}$ for $Y(s) = \frac{e^{-5s}s}{s^2 + 4}$.

Solution. First observe that $Y(s) = e^{-5s}\mathcal{L}\{\cos 2t\}$. By (10.6.4), the alternate form of the shift property in the $t$-domain,

$$e^{-5s}\mathcal{L}\{\cos 2t\} = 
\mathcal{L}\{\cos (2(t - 5))\cdot u(t - 5)\}.$$ 

Consequently,

$$\mathcal{L}^{-1}\{Y(s)\} = \cos(2t - 10)\cdot u(t - 5).$$

Since the inverse Laplace transform inherits the linearity property from the Laplace transform, it is also a linear operator as we state more formally in the next theorem.

**Linear Property of the Inverse Laplace Transform**

**Theorem 10.14.** If $F_1(s)$ and $F_2(s)$ are functions whose inverse Laplace transforms exist for $s > b$, then for any constants $c_1$ and $c_2$, an inverse transform of the linear combination $c_1F_1(s) + c_2F_2(s)$ for $s > b$ is given by

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1\mathcal{L}^{-1}\{F_1(s)\} + c_2\mathcal{L}^{-1}\{F_2(s)\}.$$ 

In other words, $\mathcal{L}^{-1}$ is a linear operator.

Example 10.46. Find $\mathcal{L}^{-1}\left\{\frac{5s}{s^2 + 9} - \frac{24}{s^4}\right\}$.

Solution. By the above linearity property,

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2 + 9} - \frac{24}{s^4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 9}\right\} - \mathcal{L}^{-1}\left\{\frac{4 \cdot 3!}{s^4}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 3^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = 5\cos 3t - 4t^3.$$
Example 10.47. Find \( \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 5} \right\} \).

Solution. First note that \( \frac{s}{s^2 + 5} = \frac{s}{s^2 + (\sqrt{5})^2} = \mathcal{L}\{\cos t \sqrt{5}\} \). Thus, by the linearity property
\[
\mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 5} \right\} = 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + (\sqrt{5})^2} \right\} = 2 \cos(t \sqrt{5}).
\]

The previous examples illustrate that a function may have to be expressed in a different form in order to determine its inverse Laplace transform. Some techniques that are useful in this regard are the same ones that aid in carrying out certain integrations, such as
- partial fraction expansion, and
- completion of the square.

Example 10.48. Find \( \mathcal{L}^{-1}\{F(s)\} \) for
\[
F(s) = \frac{26s - 13}{2s^2 + 7s - 15}.
\]

Solution. A good rule of thumb to follow is to first factor any reducible quadratic functions in the denominator. After this is done, we will know how to write the partial fraction expansion:
\[
F(s) = \frac{26s - 13}{(2s - 3)(s + 5)} = \frac{A}{2s - 3} + \frac{B}{s + 5}.
\]
Clearing the fractions of their denominators, we obtain
\[
A(s + 5) + B(2s - 3) = 26s - 13.
\]
Set \( s = -5 \) to find \( B \):
\[
-13B = 26(-5) - 13 \quad \Rightarrow \quad B = -\frac{143}{13} = 11.
\]
Set \( s = \frac{3}{2} \) to find \( A \):
\[
\frac{13}{2} A = 26 \left( \frac{3}{2} \right) - 13 \quad \Rightarrow \quad A = \frac{52}{13} = 4.
\]
And so
\[
\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1} \left\{ \frac{4}{2s - 3} \right\} + 11 \mathcal{L}^{-1} \left\{ \frac{1}{s + 5} \right\}
\]
\[
= \frac{4}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s - \frac{3}{2}} \right\} + 11 \mathcal{L}^{-1} \left\{ \frac{1}{s - (-5)} \right\} = 2e^{\frac{3t}{2}} + 11e^{-5t}.
\]
10.7. INVERSE LAPLACE TRANSFORMS

Example 10.49. Find the inverse Laplace transform of

\[ G(s) = \frac{1}{s^2 + 4s + 8}. \]

Solution. Unlike the previous example, partial fraction expansion will not work here because the quadratic function is irreducible. In a case as this, the rule of thumb is to complete the square:

\[ G(s) = \frac{1}{(s + 2)^2 + 2^2}. \]

Unfortunately, this does not quite have the form of any of the functions appearing in the right-hand column of Table 10.1. On the other hand, it is clearly related to the form \( b/(s^2 + b^2) \). All that is required is a little mathematical legerdemain:

\[ \frac{1}{(s + 2)^2 + 2^2} = \frac{1}{2} \cdot \frac{2}{(s + 2)^2 + 2^2}. \]

By linearity and the shift property in the \( s \)-domain, the inverse is

\[ \mathcal{L}^{-1}\{G(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{(s + 2)^2 + 2^2} \right\} = \frac{1}{2} e^{-2t} \sin 2t. \]

Example 10.50. Find the inverse transform of \( H(s) = \frac{3s}{s^2 + 4s + 13} \).

Solution. Again the quadratic function in the denominator is irreducible. Completing its square, we have

\[ H(s) = \frac{3s}{(s + 2)^2 + 3^2}. \]

\( H(s) \) essentially has the form \( s/(s^2 + b^2) \) except for the shift from \( s \) to \( s + 2 \) in the denominator. This calls for another trick of the mathematical trade, namely, to shift the \( s \) in the numerator:

\[ H(s) = \frac{3s}{(s + 2)^2 + 3^2} = \frac{3(s + 2) - 3 \cdot 2}{(s + 2)^2 + 3^2} = \frac{s + 2}{(s + 2)^2 + 3^2} - \frac{3}{(s + 2)^2 + 3^2}. \]

Therefore, by linearity and the shift property in the \( s \)-domain,

\[ \mathcal{L}^{-1}\{H(s)\} = 3 \mathcal{L}^{-1}\left\{ \frac{s + 2}{(s + 2)^2 + 3^2} \right\} - 2 \mathcal{L}^{-1}\left\{ \frac{3}{(s + 2)^2 + 3^2} \right\} = 3e^{-2t} \cos 3t - 2e^{-2t} \sin 3t. \]

The point of the next two examples is to demonstrate how the convolution theorem (Theorem 10.13) can be used to find inverse transforms of rational functions.
Example 10.51. Find the inverse transform of
\[ P(s) = \frac{1}{s^2 (s^2 + 1)}. \]

Solution 1. By the convolution theorem,
\[ P(s) = \frac{1}{s^2} \cdot \frac{1}{s^2 + 1} = \mathcal{L}\{t\} \mathcal{L}\{\sin t\} = \mathcal{L}\{t \ast \sin t\}. \]
So the inverse transform of \( P(s) \) is
\[ \mathcal{L}^{-1}\{P(s)\} = t \ast \sin t = \int_0^t (t - u) \sin u \, du. \]
Integrating by parts, we get
\[ \mathcal{L}^{-1}\{P(s)\} = -[t - u \cos u - \sin u]^t_0 = t - \sin t. \]

Solution 2. A partial fraction expansion for \( P(s) \) is
\[ P(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2 + 1}, \]
where \( A = 0, B = 1, \) and \( C = -1. \) Thus, So the inverse transform of \( P(s) \) is
\[ \mathcal{L}^{-1}\{P(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = t - \sin t. \]

Example 10.51 shows that the convolution theorem is an alternative to partial fraction expansion for finding inverse transforms of rational functions.

Example 10.52. Find the inverse transform of
\[ Q(s) = \frac{s}{(s^2 + 1)^2}. \]

Solution. From
\[ Q(s) = \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 1} = \mathcal{L}\{\sin t\} \mathcal{L}\{\cos t\} = \mathcal{L}\{\sin t \ast \cos t\}, \]
we have
\[ \mathcal{L}^{-1}\{Q(s)\} = \sin t \ast \cos t = \int_0^t \sin(t - u) \cos u \, du. \]

With the trigonometric identity
\[ \sin \theta \cos \phi = \frac{1}{2}[\sin(\theta - \phi) + \sin(\theta + \phi)], \]
we can rewrite the integrand as
\[\sin (t - u) \cos u = \frac{1}{2} \left[ \sin(t - 2u) + \sin(t) \right],\]

enabling us to carry out the integration:
\[
\int_0^t \sin(t - u) \cos u \, du = \frac{1}{2} \int_0^t \left[ \sin(t - 2u) + \sin(t) \right] \, du
\]
\[
= \frac{1}{4} \cos(t - 2u) \bigg|_0^t + \frac{1}{2} \sin t = \frac{1}{2} t \sin t.
\]

Therefore,
\[\mathcal{L}^{-1}\{Q(s)\} = \frac{1}{2} t \sin t.\]

### 10.8 Second-Order Linear Equations

The next theorem adds to the tools that enable us to solve second-order linear differential equations with Laplace transforms.

**Theorem 10.15.** Suppose that the function \( f(t) \) is twice differentiable on \([0, \infty)\) and both \( f(t) \) and \( f'(t) \) are of exponential order \( \gamma \). If the second derivative \( f''(t) \) is integrable on \([0, T]\) for every \( T \geq 0 \), then its Laplace transform exists for all \( s > \gamma \) and is given by the formula

\[\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0).\]  

**Proof.** Since \( f''(t) \) exists on \([0, \infty)\), both \( f(t) \) and \( f'(t) \) are continuous on \([0, \infty)\). This, along with them being of exponential order \( \gamma \), implies that their transforms exist by Corollary 10.2. Thus, by Theorem 10.3,

\[\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).\]  

Again appealing to Theorem 10.3, we see that the Laplace transform of \( f''(t) \) exists by the integrability of \( f''(t) \) on \([0, T]\) for every \( T \geq 0 \) and by \( f'(t) \) being differentiable and of exponential order \( \gamma \); hence,

\[\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0).\]

Using (10.8.2), we obtain

\[\mathcal{L}\{f''(t)\} = s\left[ s\mathcal{L}\{f(t)\} - f(0) \right] - f'(0),\]

which simplifies to (10.8.1).
The following examples illustrate Theorem 10.15 and the properties in Section 10.6. We list the primary properties or techniques in boldface that are used with each of the examples.

Example 10.53. (linear factors; partial fraction expansion)

Solve the initial value problem

\[ x'' + 6x' + 5x = 36e^t, \quad x(0) = 0, \quad x'(0) = 10. \]

Solution. First take the Laplace transform of both sides of the differential equation and use the linearity property:

\[ \mathcal{L}\{x''(t)\} + 6\mathcal{L}\{x'(t)\} + 5\mathcal{L}\{x(t)\} = 36\mathcal{L}\{e^t\}. \]

Let \( X(s) := \mathcal{L}\{x(t)\} \). Then use Table 10.1 and apply Theorem 10.3 and Theorem 10.15:

\[ s^2X(s) - sx(0) - x'(0) + 6[sX(s) - x(0)] + 5X(s) = 36 \cdot \frac{1}{s - 1}. \]

Now use the initial conditions and collect like terms:

\[ (s^2 + 6s + 5)X(s) - s \cdot 0 - 10 = 36 \cdot \frac{1}{s - 1}. \]

Then solve for \( X(s) \):

\[ X(s) = \frac{36}{s - 1} + \frac{10}{(s + 1)(s + 5)} = \frac{36 + 10(s - 1)}{(s - 1)(s + 1)(s + 5)} = \frac{10s + 26}{(s - 1)(s + 1)(s + 5)}. \]

Expand \( X(s) \) in partial fractions:

\[ \frac{10s + 26}{(s - 1)(s + 1)(s + 5)} = \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{C}{s + 5}. \]

Clearing the fractions of their denominators, we have

\[ A(s + 1)(s + 5) + B(s - 1)(s + 5) + C(s - 1)(s + 1) = 10s + 26. \]

Now we can easily find \( A \) by letting \( s = 1 \):

\[ (2)(6)A = 10(1) + 26 \quad \Rightarrow \quad 12A = 36 \quad \Rightarrow \quad A = 3. \]

To find \( B \), let \( s = -1 \):

\[ (-2)(4)B = 10(-1) + 26 \quad \Rightarrow \quad B = -2. \]

Finally, let \( s = -5 \):

\[ (-6)(-4)C = 10(-5) + 26 \quad \Rightarrow \quad C = -1. \]
And so the Laplace transform is
\[ X(s) = \frac{3}{s - 1} - \frac{2}{s + 1} - \frac{1}{s + 5}. \]
Taking the inverse Laplace transform of \( X(s) \), we arrive at the solution
\[ x(t) = L^{-1}\{X(s)\} = 3e^t - 2e^{-t} - e^{-5t}. \]

Example 10.54. (quadratic factors; partial fraction expansion; shift in the \( s \)-domain)
Solve the initial value problem
\[ x'' + 2x' + 10x = -25e^{3t}, \quad x(0) = 0, \quad x'(0) = 6. \]

Solution. Taking the Laplace transform of both sides of the differential equation, we get
\[ \mathcal{L}\{x''(t)\} + 2\mathcal{L}\{x'(t)\} + 10\mathcal{L}\{x(t)\} = -25\mathcal{L}\{e^{3t}\}. \]
Letting \( X(s) = \mathcal{L}\{x(t)\} \), using Table 10.1, and applying the transforms of derivatives properties, this becomes
\[ s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 10X(s) = -25 \cdot \frac{1}{s - 3}. \]
Now solve for \( X(s) \):
\[ X(s) = \frac{6s - 43}{(s - 3)[(s + 1)^2 + 9]}. \]
The partial fraction expansion of \( X(s) \) is
\[ \frac{6s - 43}{(s - 3)[(s + 1)^2 + 9]} = \frac{A}{s - 3} + \frac{B(s + 1) + 3C}{(s + 1)^2 + 9}. \]
Clearing the fractions of their denominators, we have
\[ A[(s + 1)^2 + 9] + [B(s + 1) + 3C](s - 3) = 6s - 43. \]
Setting \( s \) equal to 3, -1, and 0, we find \( A = -1 \), \( B = 1 \), and \( C = 10/3 \). Thus,
\[ X(s) = -\frac{1}{s - 3} + \frac{s + 1}{(s + 1)^2 + 3^2} + \frac{10}{3} \cdot \left[ \frac{3}{(s + 1)^2 + 3^2} \right]. \]
Therefore, the solution is given by the inverse Laplace transform of \( X(s) \):
\[ x(t) = -e^{3t} + e^{-t} \cos 3t + \frac{10}{3} e^{-t} \sin 3t. \]
Example 10.55. (both shift properties)

Solve the initial value problem

\[ x'' + 6x' + 13x = 5\delta(t - 2), \quad x(0) = 0, \quad x'(0) = 2. \]

**Solution.** As is always the case, the first step is to take the Laplace transform of the equation:

\[ \mathcal{L}\{x''(t)\} + 6\mathcal{L}\{x'(t)\} + 13\mathcal{L}\{x(t)\} = 5\mathcal{L}\{\delta(t - 2)\}. \]

Applying the transforms of derivatives properties and using Table 10.1, we find that

\[ s^2X(s) - sx(0) - x'(0) + 6[sX(s) - x(0)] + 13X(s) = 5e^{-2s} \]

where \( X(s) = \mathcal{L}\{x(t)\} \). Thus,

\[ (s^2 + 6s + 13)X(s) = 2 + 5e^{-2s}. \]

Solving for \( X(s) \), we have

\[ X(s) = \frac{2}{s^2 + 6s + 13} + \frac{5e^{-2s}}{s^2 + 6s + 13}. \]

Observing that \( s^2 + 6s + 13 \) is irreducible over the reals, we complete the square obtaining

\[ X(s) = \frac{2}{(s + 3)^2 + 2^2} + \frac{5e^{-2s}}{(s + 3)^2 + 2^2}. \]

To find the solution, we must take the inverse Laplace transform of \( X(s) \). It is clear that

\[ \frac{2}{s^2 + 2^2} = \mathcal{L}\{\sin 2t\} \]

is involved. By the shift property in the \( s \)-domain,

\[ \mathcal{L}\{e^{-3t}\sin 2t\} = F(s - (-3)) = F(s + 3) = \frac{2}{(s + 3)^2 + 2^2}. \]

To find the inverse transform of the second term, we apply the shift property in the \( t \)-domain:

\[ e^{-bs}\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t - b)u(t - b)\}. \]

With \( b = 2 \), we have

\[ e^{-2s}\cdot \frac{2}{(s + 3)^2 + 2^2} = e^{-2s}\mathcal{L}\{e^{-3t}\sin 2t\} \]

\[ = \mathcal{L}\{f(t - 2)u(t - 2)\} = \mathcal{L}\{e^{-3(t-2)}\sin [2(t-2)]u(t-2)\}, \]

where \( f(t) = e^{-3t}\sin 2t \). And so

\[ \mathcal{L}\left\{e^{-2s}\cdot \frac{2}{(s + 3)^2 + 2^2}\right\} = e^{6-3t}\sin(2t-4)u(t-2). \]
Therefore, the solution is
\[ x(t) = e^{-3t} \sin 2t + \frac{5}{2} e^{6-3t} \sin (2t - 4)u(t - 2) \]
\[ = \begin{cases} 
  e^{-3t} \sin 2t, & \text{if } t < 2 \\
  e^{-3t} \sin 2t + \frac{5}{2} e^{6-3t} \sin (2t - 4), & \text{if } t > 2.
\end{cases} \]

**Example 10.56. (convolution theorem)**

Find a formula for the solution of the initial value problem
\[ x'' + 4x = g(t), \quad x(0) = 0, \quad x'(0) = 2. \]

Assume \( g(t) \) is piecewise continuous on \([0, \infty)\) and of exponential order.

**Solution.** Taking the Laplace transform of the differential equation, we get
\[ s^2 X(s) - sx(0) - x'(0) + 4X(s) = G(s), \]
where \( X(s) = \mathcal{L}\{x(t)\} \) and \( G(s) = \mathcal{L}\{g(t)\} \). Because of the initial conditions, this becomes
\[ (s^2 + 4)X(s) - s \cdot 0 - 2 = G(s) \]
and so
\[ X(s) = \frac{2}{s^2 + 4} + \frac{1}{s^2 + 4} \cdot G(s). \]
That is,
\[ \mathcal{L}\{x(t)\} = \mathcal{L}\{\sin 2t\} + \frac{1}{2} \mathcal{L}\{\sin 2t \ast g(t)\}. \]
By the convolution theorem,
\[ \mathcal{L}\{x(t)\} = \mathcal{L}\{\sin 2t\} + \frac{1}{2} \mathcal{L}\{2t \ast g(t)\}. \]
Consequently, the solution is
\[ x(t) = \sin 2t + \frac{1}{2} \int_0^t \sin[2(t - u)]g(u) \, du. \]

### 10.8.1 Equations with Variable Coefficients

Differential equations with variable coefficients are usually insolvable with Laplace transforms. Nevertheless there are exceptions as the following examples will reveal. The Laplace transform of a second-order linear equation with coefficients that are constants or polynomials of the first degree (but no higher) is a first-order linear equation. The reason is due to the formulas below, which we derive using the derivative of a
transform property (Theorem 10.8) and the transform of derivatives properties (Theorems 10.3 and 10.15). For \( X(s) = \mathcal{L}\{x(t)\} \),

\[
\mathcal{L}\{tx(t)\} = -\frac{d}{ds}\mathcal{L}\{x(t)\} = -X'(s), \tag{10.8.3}
\]

\[
\mathcal{L}\{tx'(t)\} = -\frac{d}{ds}\mathcal{L}\{x'(t)\} = -\frac{d}{ds}[sX(s) - x(0)] = -sX'(s) - X(s), \tag{10.8.4}
\]

and

\[
\mathcal{L}\{tx''(t)\} = -\frac{d}{ds}\mathcal{L}\{x''(t)\} = -\frac{d}{ds}[s^2X(s) - sx(0) - x'(0)] \tag{10.8.5}
\]

\[= -s^2X'(s) - 2sX(s) + x(0).\]

**Example 10.57.** Show that the transform of the second-order linear equation

\[tx'' + 2x' + tx = f(t)\]

is a first-order linear equation. Assume the transform of \( f(t) \) exists.

**Solution.** By (10.8.3) and (10.8.5), the Laplace transform of the differential equation is

\[-s^2X'(s) - 2sX(s) + x(0) + 2[sX(s) - x(0)] - X'(s) = F(s),\]

where \( F(s) := \mathcal{L}\{f(t)\} \). Simplifying we obtain the first-order equation

\[X'(s) = -\frac{F(s)}{s^2 + 1} - \frac{x(0)}{s^2 + 1}. \tag{10.8.6}\]

**Example 10.58.** Solve the initial value problem

\[tx'' + 2x' + tx = 0, \quad x(0) = 1, \quad x'(0) = 0. \tag{10.8.7}\]

**Solution 1.** For \( x(0) = 1 \) and \( \mathcal{L}\{0\} = 0 , (10.8.6) \) is

\[X'(s) = -\frac{1}{s^2 + 1}. \tag{10.8.8}\]

Thus,

\[-\frac{d}{ds}X(s) = \frac{1}{s^2 + 1} = \mathcal{L}\{\sin t\}.\]

By the derivative of a transform property,

\[-\frac{d}{ds}X(s) = -\frac{d}{ds}\mathcal{L}\{x(t)\} = \mathcal{L}\{tx(t)\}.\]
10.8. SECOND-ORDER LINEAR EQUATIONS

Consequently,
\[ \mathcal{L}\{tx(t)\} = \mathcal{L}\{\sin t\}. \]

Therefore,
\[ tx(t) = \sin t. \]

At first glance, it seems that we are done and
\[ x(t) = \frac{\sin t}{t} \]
is the solution. However, there is the initial condition \( x(0) = 0 \) but \( \sin t/t \) is not defined at \( t = 0 \). But it does have a removable discontinuity at \( t = 0 \) as
\[ \lim_{t \to 0^+} \frac{\sin t}{t} = 1. \]

Therefore,
\[ x(t) = \begin{cases} \frac{\sin t}{t}, & t > 0 \\ 1, & t = 0 \end{cases} \quad (10.8.9) \]
is a continuous solution of (10.8.7) for \( t \geq 0 \).

**Solution 2.** Another approach is to first solve the transformed equation (10.8.8):
\[ X(s) = -\int \frac{ds}{s^2 + 1} = -\tan^{-1} s + C. \]

Taking the limit of \( X(s) \) as \( s \to \infty \), we have
\[ \lim_{s \to \infty} X(s) = -\frac{\pi}{2} + C. \]

On the other hand, assuming \( X(s) \) is the transform of a function that is piecewise continuous on \([0, \infty)\) and of exponential order, its long-term behavior is
\[ \lim_{s \to \infty} X(s) = 0 \]
by Corollary 10.1. Thus,
\[ C = \frac{\pi}{2}. \]

As a result,
\[ X(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}. \]

By (10.6.13),
\[ \mathcal{L}^{-1} \left\{ \tan^{-1} \frac{1}{s} \right\} = \frac{\sin t}{t}. \]

Therefore, as argued earlier, (10.8.9) is the continuous solution for \( t \geq 0 \).
Example 10.59. Solve the initial value problem
\[ x'' + tx' - 2x = 1, \quad x(0) = x'(0) = 0. \] (10.8.10)

Solution. The transformed equation is the linear first-order equation
\[ s^2 X(s) - s x(0) - x'(0) - s X'(s) - X(s) - 2 X(s) = \frac{1}{s}. \]

See (10.8.4). Using the initial conditions and rearranging terms to write this in standard form, we obtain
\[ X'(s) - \left( \frac{3}{s} - s \right) X(s) = -\frac{1}{s^2}. \] (10.8.11)

An integrating factor for (10.8.11) is
\[ \mu(s) = e^{\int \left( \frac{3}{s} - s \right) ds} = s^3 e^{-s^2/2}. \]

Multiplying (10.8.11) by \[ \mu(s) \] and integrating, we obtain
\[ \frac{d}{ds} \left[ s^3 e^{-s^2/2} X(s) \right] = -s e^{-s^2/2} \]
\[ s^3 e^{-s^2/2} X(s) = -\int s e^{-s^2/2} ds + C = -e^{-s^2/2} + C. \]

Thus,
\[ X(s) = \frac{1}{s^3} + C \frac{e^{s^2/2}}{s^3}. \]

By Corollary 10.1,
\[ \lim_{s \to \infty} X(s) = 0, \]
assuming \[ X(s) \] is the transform of a piecewise continuous function of exponential order. But this can only be the case if \[ C = 0 \] as
\[ \lim_{s \to \infty} \frac{e^{s^2/2}}{s^3} = \infty. \]

Therefore,
\[ X(s) = \frac{1}{s^3}. \]

Consequently, the solution is
\[ x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2!}{s^3} \right\} = \frac{1}{2} t^2. \]

A quick check verifies that this does solve the differential equation and satisfy the initial conditions of the initial value problem (10.8.10).
Problems

You see, in every job that must be done, there is an element of fun. You find the fun and snap! The job’s a game. 

from Walt Disney’s 1964 film Mary Poppins
(Mary Poppins’ advice to Jane and Michael)

Laplace Transforms from the Definition

Use Definition 10.1 on page 317 to find the Laplace transforms of the functions in Problems 1 through 6.

1. \( \cos bt \), where \( b \) is any real constant
2. \( f(t) = \begin{cases} e^{2t} & \text{if } t < 5 \\ 0 & \text{if } t > 5 \end{cases} \)
3. \( g(t) = \begin{cases} 0 & \text{if } t < 5 \\ e^{2t} & \text{if } t > 5 \end{cases} \)
4. \( \sinh t := (e^t - e^{-t})/2 \) (the hyperbolic sine function)
5. \( h(t) = \begin{cases} 1 - t & \text{if } t < 7 \\ 10 & \text{if } t > 7 \end{cases} \)
6. \( tu(t - 2) \)

Linearity of the Laplace Transform

Use Table 10.1 and Theorem 10.1 to find the Laplace transforms of the functions in Problems 7 through 12.

7. \( \sin 2t + e^{6t} \)
8. \( 3e^{-2t} + 2t^7 - 12 \)
9. \( 5u(t - 2) \)
10. \( \delta(t - 2) \)
11. \( \cosh (t/2) \), where \( \cosh t := (e^t + e^{-t})/2 \)
12. \( \frac{3}{2}\delta(t - 5) - 2 \cos \frac{3}{2}t \)

Equivalent Properties

13. Show that the linearity property (10.3.1) is equivalent to the following two properties:
   (a) \( \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \)
   (b) \( \mathcal{L}\{cf(t)\} = c\mathcal{L}\{g(t)\} \),

where \( c \) is a constant. In other words, show that (10.3.1) implies (a) and (b) and that conversely (a) and (b) imply (10.3.1).

Properties of the Laplace Transform

In Problems 14 through 30, determine the Laplace transform of each of the following functions using Table 10.1 and the properties of the transform.

14. \( e^{5t}t^3 \)
15. \( e^{-2t} \sin \frac{t}{2} \)
16. \( 10t^4 - \sin 2t \)
17. \( t \cos 2t \)
18. \( 3t^4 - \sin 2t \)
19. \( t^5 e^{-t} \)
20. \( 1 + \frac{3}{2}e^{-5t} - \delta(t - 3) \)
21. \( t e^{2t} \sin 3t \)
22. \( t^2 \sin t \)
23. \( 2 \cos \frac{t}{2} + \begin{cases} 0 & \text{if } t < 5 \\ 10 & \text{if } t > 5 \end{cases} \)
24. \( t \sin 5t + 4(t - 1)^2 \)
25. \( t^{10} + 3\delta(t - 2) \)
26. \( 5 + 8t \sin \frac{t}{2} - \cosh 5t \)
27. \( h(t) = \begin{cases} 3 + e^{5t} & \text{if } t < 1 \\ e^{5t} & \text{if } t > 1 \end{cases} \)
28. \( \frac{1}{15}t^2 \sin 7t - 3t^4 e^{-2t} + tu(t - 4) \)
29. \( \cos(3t)u(t - \pi/6) \)
30. \( \sin(3t + 2) \)

Potpourri

31. Explain why \( \cos bt \) is of exponential order 0.
32. Explain why \( \sin bt = O(1) \) as \( t \to \infty \).
33. Explain why \( e^t \) is of exponential order 1.
34. Show that \( f(t) = O(e^{\gamma t}) \) as \( t \to \infty \) is equivalent to the statement: \( e^{-\gamma t} |f(t)| \) is bounded for \( t \geq t_1 \) for some \( t_1 \geq 0 \).
35. Use the Laplace Transform of the Derivative Property to find \( \mathcal{L}\{\cos bt\} \) from
\[
\mathcal{L}\{\cos bt\} = \frac{b}{s^2 + b^2}.
\]
36. Find \( \mathcal{L}\{\sin^2 bt\} \) by using an appropriate half-angle identity. (If you do not know this identity, look it up in any trigonometry book, find out how it is derived, and memorize it.) Compare your answer with the result obtained in one of the examples.
37. Let \( f(t) \) be continuous on \([0, \infty)\) and of exponential order \( \gamma \geq 0 \). Generalize the transform of an integral property by determining the transform of \( \int_a^t f(\tau) \, d\tau \), where \( a \geq 0 \).
38. Suppose \( f(t) \) is continuous on \([0, \infty)\) and of exponential order \( \gamma \). If \( \mathcal{L}\{f(t)\} = 0 \) for \( s > \gamma \), show that \( \mathcal{L}\{f(t)P(t)\} = 0 \) for \( s > \gamma \) for all polynomials \( P(t) \).

### First-Order Linear Equations

In Problems 39 through 47, solve the initial value problems using the Method of Laplace Transforms. Then check your answers by solving with integrating factors.

39. \( x' - 5x = e^{2t} \), \( x(0) = 1 \)
40. \( y' + 2y = 4e^{-3t} \), \( y(0) = -2 \)
41. \( y' + \frac{1}{2}y = 0 \), \( y(0) = 4 \)
42. \( x' + 3x = \cos 2t \), \( x(0) = 1 \)
43. \( x' - 2x = 10 \cos t \), \( x(0) = 3 \)
44. \( x' - 10x = 29 \cos 5t \), \( x(0) = 1 \)
45. \( x' + x = 8u(t - 10) \), \( x(0) = -5 \)
46. \( x' - 3x = 6\delta(t - 5) \), \( x(0) = 2 \)
47. \( x' - x = 4\delta(t - 5) \), \( x(0) = -8 \)

### Inverse Laplace Transforms

Find the inverse Laplace transform of each of the following functions.

48. \( \frac{3}{s^2 + 2} \)
49. \( \frac{s + 2}{s^2 + 4s - 5} \)
50. \( \frac{2s - 4}{s^2 - 4s + 18} \)
51. \( \frac{s}{s^2 + 4s + 20} \)
52. \( \frac{5s^2 + 3s + 39}{(s - 2)(s^2 + 9)} \)
53. \( \frac{8}{(s - 2)^3} \)
54. \( \frac{12s}{4s^2 + 8s + 5} \)
55. \( \frac{7}{s - 5} - \frac{2s}{s^2 + 16} \)
56. \( \frac{5s}{s + 4} - \frac{s^2 + 7}{s^2 + 6} \)
57. \( \frac{2s^2 - s + 5}{s^3 + s^2 - 6s} \)

### Second-Order Linear Equations

Use the method of Laplace Transforms to solve the following linear second-order initial value problems.

58. \( \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} = 8 \), \( x(0) = 0 \), \( x'(0) = 0 \)
59. \( \ddot{x} - 4\dot{x} + 5x = 6e^{3t} \), \( x(0) = 2 \), \( \dot{x}(0) = -5 \)
60. \( x'' + 9x = 2e^t \), \( x(0) = 1 \), \( x'(0) = 0 \)
61. \( \frac{d^2x}{dt^2} + 5 \frac{dx}{dt} - 14x = 0 \), \( x(0) = 5 \), \( x'(0) = 1 \)
62. \( \ddot{x} + 2\dot{x} + 10x = \frac{1}{2}t^2 - 5e^{3t} \), \( x(0) = 0 \), \( \dot{x}(0) = 6 \)
63. \( x'' + 4x' + 5x = 6\delta(t - 2) \), \( x(0) = 0 \), \( x'(0) = 1 \)
64. \( \ddot{x} + 4x = 1 - 2u(t - 5) \),
\[ x(0) = -1, \quad \dot{x}(0) = 0 \]