

Ordinary Differential Equations

Concepts, Methods, and Models

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Bear in mind that the wonderful things you learn in your school are the work of many generations, produced by enthusiastic effort and infinite labor in every country of the world. All this is put into your hands as your inheritance in order that you may receive it, honor it, add to it, and one day faithfully hand it on to your children. Thus do we mortals achieve immortality in the permanent things which we create in common.

Albert Einstein¹

¹*Ideas and Opinions* [23, p. 56]. The original source is *Mein Weltbild* [24].

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