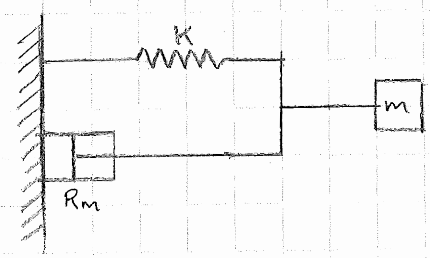
Damped Oscillations

Unfortunately, in real life, no system is 100% free of loss, so we'll need to account for this in our model. Sliding friction and fluid drag are two examples. Since we're worried about how a body vibrates in air (a fluid), let's focus on the second one. For relatively low speeds, it can be shown that fluid drag is roughly proportional to the velocity of the vibrating body. That is, the drag force F_r can be written as

$$F_r = -R_m \frac{dx}{dt}$$

where R_m is the mechanical resistance of the system. In our mass-spring system, we might model this friction with a dashpot, a device used in shock absorbers and other mechanisms to introduce damping:



So by Newton's Second Law:

$$\Sigma F = ma = -R_m \frac{dx}{dt} - Kx$$

or

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + Kx = 0$$

or

$$\frac{d^2x}{dt^2} + \frac{R_m}{m}\frac{dx}{dt} + \frac{K}{m}x = 0$$

For convenience and comparison, we will again use $\omega_0 = \sqrt{\frac{K}{m}}$

We will also define a parameter $\alpha = \frac{R_m}{2m}$, which we call the <u>decay constant</u>, for reasons to be seen later.

This leaves our equation of motion in the form

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = 0$$

We can solve this using our phasor method, i.e. solving the differential equation

$$\frac{d^2\tilde{x}}{dt^2} + 2\alpha\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = 0$$

and taking the real part. Our guess will be a complex exponential,

$$\hat{x} = \tilde{A}e^{\gamma t}$$

where γ is a general variable that we will find in terms of our system's parameters. For organization reasons, i is included in γ.

If we substitute our guess into the differential equation, we get

$$\gamma^2 \tilde{A} e^{\gamma t} + 2\alpha\gamma \tilde{A} e^{\gamma t} + \omega_0^2 \tilde{A} e^{\gamma t} = 0$$

or

$$(\gamma^2 + 2\alpha\gamma + \omega_0^2)\tilde{A} e^{\gamma t} = 0$$

Since an exponential can never equal 0, this requires ($\tilde{A}=0$ is trivial)

$$\gamma^2 + 2\alpha\gamma + \omega_0^2 = 0$$

or

$$\gamma_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The overall solution depends on the second term in this expression, and in particular, the relationship between α and ω_0, since this determines the sign of the expression under the radical.
We have 3 cases to consider:

1) $\alpha > \omega_0$

If $\alpha > \omega_0$, then $\alpha^2 > \omega_0^2$, so $\alpha^2 - \omega_0^2 > 0$, meaning γ_1 and γ_2 are both real, so our solution is of the form

$$x(t) = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}$$
$$= e^{-\alpha t}\left(c_1 e^{\sqrt{\alpha^2-\omega_0^2}\,t} + c_2 e^{-\sqrt{\alpha^2-\omega_0^2}\,t}\right)$$

This is already purely real, so we don't need to take the real part. It is a decaying exponential with no oscillation. We call this an overdamped system, since it has such large damping that no oscillation can occur.

2) $\alpha = \omega_0$

If $\alpha = \omega_0$, $\sqrt{\alpha^2 - \omega_0^2} = 0$, so we have a double root:

$$\gamma_1 = \gamma_2 = -\alpha$$

So our solution has the form

$$x(t) = (c_1 + c_2 t)e^{-\alpha t}$$

Which is another exponentially decaying system with no oscillation. It is called a critically damped system, since it is the dividing line between oscillation and no oscillation.

3) $\alpha < \omega_0$

If $\alpha < \omega_0$, then $\alpha^2 - \omega_0^2 < 0$, so $\sqrt{\alpha^2 - \omega_0^2}$ is imaginary. To extract the real of this term, we will write $\gamma_{1,2}$ as

$$\gamma_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm i\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm i\omega_d, \text{ where } \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

So our complex solution is

$$\tilde{x}(t) = \tilde{A}_1 e^{\gamma_1 t} + \tilde{A}_2 e^{\gamma_2 t} = e^{-\alpha t}\left(\tilde{A}_1 e^{i\omega_d t} + \tilde{A}_2 e^{-i\omega_d t}\right)$$

The real part of this gives our physical solution $x(t)$:

$$x(t) = \mathrm{Re}(\tilde{x}(t)) = Ae^{-\alpha t}\cos(\omega_d t + \phi)$$

So we see that this solution <u>does</u> oscillate, but with a decaying exponential "envelope." Since this is the only one of the three solutions that has oscillations, it is the one typically considered in acoustics. It is called an <u>underdamped</u> system.

Let's find our full solution by applying the initial conditions $x(0) = x_0$ and $v(0) = v_0$. We get:

$$x(t) = e^{-\alpha t}\sqrt{x_0^2 + \left(\frac{v_0+\alpha x_0}{\omega_d}\right)^2}\cos\left(\omega_d t - \tan^{-1}\left(\frac{v_0+\alpha x_0}{\omega_d x_0}\right)\right).$$

Although this is not strictly a periodic function (since its amplitude is not constant), we see that the oscillatory part has an angular frequency of ω_d (the damped frequency), so we can still define a "period" T_d for the motion.

$$T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \alpha^2}}$$

This is the time between successive zeroes or between successive maxima and minima. However, these points aren't exactly halfway between zeroes now.

It should also be noted that ω_d is less than ω_0, and this explains the o subscript for ω_0. ω_0 is the natural angular frequency of the <u>undamped</u> harmonic oscillator.

One way to measure how large our damping is is to see the time τ it takes for the oscillator's amplitude to decay to $\frac{1}{e}$ of its initial value x_0. That is, when does

$$x_0 e^{-\alpha\tau} = \frac{x_0}{e} = x_0 e^{-1}$$

We see that it's when $\alpha\tau = 1$, or $\tau = \frac{1}{\alpha} = \frac{2m}{R_m}$

One thing to note is that $e^{-\alpha t}$ never equals 0, so our model says the mass will oscillate forever at smaller and smaller amplitudes. Obviously this isn't true, so we typically define a "cutoff" time of 5τ at which we can consider the time at which the mass has effectively stopped. ($\frac{1}{e^5} = .00674 \approx .7\%$ of x_0)

Now how much energy is actually lost in a DHO?

$$\frac{d}{dt}(E) = \frac{d}{dt}(E_p + E_k) = \frac{d}{dt}\left(\frac{1}{2}Kx^2 + \frac{1}{2}m\dot{x}^2\right) = Kx\frac{dx}{dt} + m\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}$$

$$= \frac{dx}{dt}\left(Kx + m\frac{d^2x}{dt^2}\right) = \frac{dx}{dt}\left(-R_m\frac{dx}{dt}\right) = -R\left(\frac{dx}{dt}\right)^2 = -2\alpha m v^2$$

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + Kx = 0$$

So energy loss is proportional to the square of the velocity.