Taylor Series Review

Remember that the goal of Taylor Series is to approximate "messier" functions with polynomials, which are relatively easy to understand, and the more terms we have, the better an approximation we get. We also notice that if we "center" this approximation around the values we want to approximate the function at, we get more accurate results. As an equation, for a function $f(x)$ that we want to approximate at center c, we need to find the coefficients $a_{0,1,2,\ldots,n}$ such that

$$f(x) \approx a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^k$$

Where the sum converges to $f(x)$ as we add more terms. With infinitely many terms, the left and right side are equal, so we can say

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$$

But how do we find these a's?

Writing out the sum, we see that

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

So if we evaluate both sides at c, we get

$$f(c) = a_0 + a_1\cancel{(c-c)}^{0} + a_2\cancel{(c-c)}^{0\,2} + \ldots$$

or $\quad f(c) = a_0$

So we have found a_0! What about a_1? If we differentiate both sides, we get

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3$$

So, again evaluating at c, we get

$$f'(c) = a_1 + \cancel{2a_2(c-c)}^{0} + \cancel{3a_3(c-c)^2}^{0} + \cancel{4a_4(c-c)^3}^{0}$$

So $f'(c) = a_1$, i.e. we have found a_1. What about a_2?

Differentiate again!

$$f''(x) = 2a_2 + 2\cdot 3\cdot a_3(x-c) + 3\cdot 4\cdot a_4(x-c)^2$$

So $\quad f''(c) = 2a_2$

Or $\quad a_2 = \dfrac{f''(c)}{2}$

If we keep going, differentiating and evaluating at c, we find

$$a_3 = \frac{f'''(c)}{2\cdot 3}$$

$$a_4 = \frac{f''''(c)}{2\cdot 3\cdot 4}$$

Or in general, $\quad a_k = \dfrac{1}{k!} f^{(k)}(c)$

So for a (well-behaved) function $f(x)$, we can write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

And approximate f by taking the first few terms of this series!

Ex: $\sin(x)$, expanded around 0. (i.e. approximating $\sin(x)$ near 0)

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
$\vdots$	$\vdots$	$\vdots$

So

$$\sin(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{(-1)}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \dots$$
$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Similarly,

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

and

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

This is where we get the small-angle approximations from!

For instance,

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots$$

If θ is small, (near zero), not only will keeping just a few terms give an accurate result, the first term will have a much bigger contribution than all the rest (e.g. $\theta^3 \ll \theta$), so we won't do much harm by only keeping the first term. This gives us, for small θ, that:

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 1$$