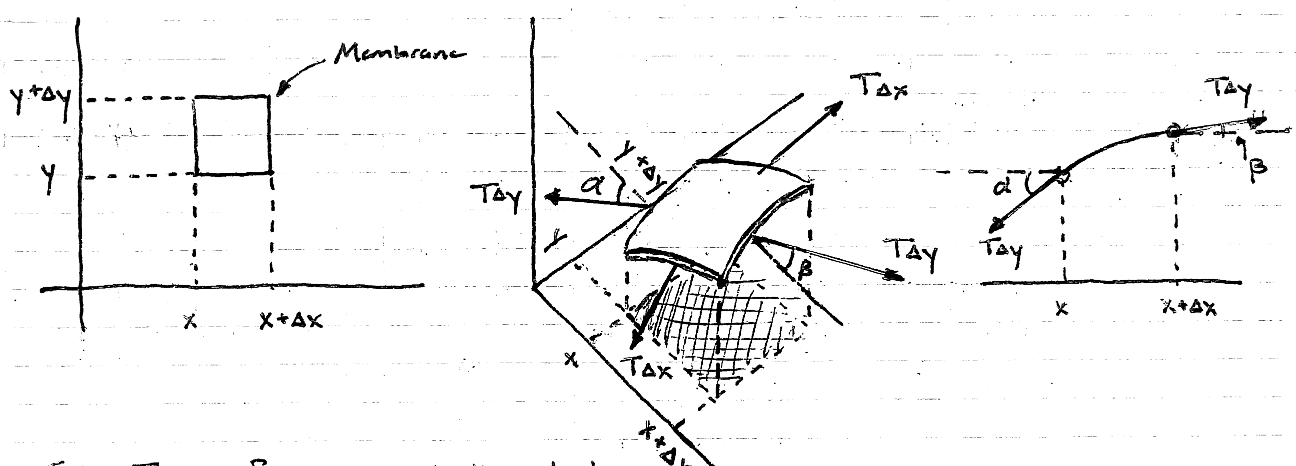
The Drum (Thin Circular Membrane)

Consider a circular drum whose skin has area density (mass per unit area) ρ. If the boundary is under a uniform tension T, the whole surface is under this same uniform tension (why?)
This tension is measured in force per unit distance.
To begin, we parametrise the surface with variables x and y and use u to denote the displacement perpendicular to the surface.
We will assume the deflection $u(x,y,t)$ is small compared to the size of the membrane, and that the angles of deflection are small.

At any time t, since the deflections/angles are small, the sides of a small portion of the membrane are approximately Δx and Δy.



Since T is the force per unit length and it is uniform, the forces acting along the sides of the membrane are $\approx T\Delta x$ and $T\Delta y$.
These forces are tangent to the membrane at every instant since it is perfectly flexible.

<u>Horizontal Force Components</u>
The horizontal components in each direction are $T\times$ the cosine of the angle of inclination.
But since all of these angles are assumed to be small, their cosines are close to 1.
Therefore, horizontal force components on opposite sides of the membrane essentially cancel.

<u>Vertical Force Components</u>
The components along the right and left side of the membrane portion are

$$T\Delta y\sin\beta \text{ and } -T\Delta y\sin\alpha$$

$$\text{So, } \Sigma F_x = T\Delta y\sin\beta - T\Delta y\sin\alpha = T\Delta y(\sin\beta-\sin\alpha) \overset{\text{Small-angle}}{\approx} T\Delta y(\tan\beta-\tan\alpha)$$

$$\approx T\Delta y\left[\frac{\partial u}{\partial x}(x+\Delta x, y_1) - \frac{\partial u}{\partial x}(x, y_1)\right]$$

We can make a similar argument for the other direction, giving

$$\Sigma F_y \approx T\Delta x\left[\frac{\partial u}{\partial y}(x_1, y+\Delta y) - \frac{\partial u}{\partial y}(x_1, y)\right]$$

Now remember, the mass per unit area of the plate is ρ, so the mass of the membrane portion (with undeflected area ΔA), is

$$m = \rho\Delta A = \rho\Delta x\Delta y$$

Therefore, Newton's 2nd Law ($\Sigma F = ma$), gives

$$\rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2} = T\Delta y\left[\frac{\partial u}{\partial x}(x+\Delta x, y_1) - \frac{\partial u}{\partial x}(x, y_1)\right] + T\Delta x\left[\frac{\partial u}{\partial y}(x_1, y+\Delta y) - \frac{\partial u}{\partial y}(x_1, y)\right]$$

Dividing both sides by $\rho \Delta x \Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{\frac{\partial u}{\partial x}(x+\Delta x, y_1) - \frac{\partial u}{\partial x}(x, y_1)}{\Delta x} + \frac{\frac{\partial u}{\partial y}(x_1, y+\Delta y) - \frac{\partial u}{\partial y}(x_1, y)}{\Delta y}\right]$$

Letting $\Delta x, \Delta y \to 0$ gives

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \text{ where } c = \sqrt{\frac{T}{\rho}}$$

This equation is called the two-dimensional wave equation. For obvious reasons, c will again end up being the speed of the waves on the membrane. Since drums are circular, they have radial symmetry, so working in polar coordinates will be simpler than Cartesian.

We can show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$, so

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

We're again going to look for separable solutions, i.e. $u(x,y,t) = u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

We get:

$$R(r)\Theta(\theta)T''(t) = c^2\left(R''(r)\Theta(\theta)T(t) + \frac{1}{r}R'(r)\Theta(\theta)T(t) + \frac{1}{r^2}R(r)\Theta''(\theta)T(t)\right)$$

Dividing both sides by $R(r)\Theta(\theta)T(t)$ gives

$$\frac{T''(t)}{T(t)} = c^2\left(\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}\right)$$

The left only depends on t, and the right only on r and θ, so both must equal some constant that we may show must be a negative number, which we'll call $-\omega^2$, giving us two differential equations

$$T''(t) = -\omega^2 T(t)$$

and

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{-\omega^2}{c^2}$$

The solution of the first is a multiple of $\sin(\omega t + \phi)$, where ϕ is determined by initial conditions. Now, if we multiply the second equation by r^2 and rearranging gives

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\omega^2}{c^2}r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

The left depends only on r, the right only on θ, so both again equal a constant which we will call n^2. Then we have

$$\Theta''(\theta) = -n^2\,\Theta(\theta) \longrightarrow \Theta(\theta) \text{ is a multiple of } \sin(n\theta + \psi)$$

and

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\omega^2}{c^2}r^2 = n^2$$

If we multiply both sides by $R(r)$ and divide by r^2, then rearrange, we get

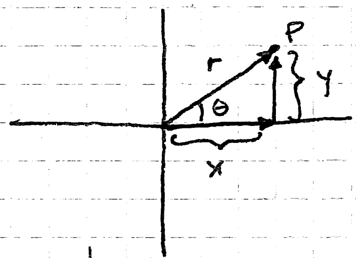
$$R''(r) + \frac{1}{r}R'(r) + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2}\right)R(r) = 0$$ This is called the Bessel equation

We first check that the Bessel equation satisfies the conditions for the Frobenius method.

1. We see that if we let $b(r) = 1$ and $c(r) = \frac{\omega^2 r^2 - n^2 c^2}{c^2}$, we have the proper form.

The 2D-Wave Equation, Cartesian → Cylindrical

Remember that we can specify a point in $\mathbb{R}^2$ either by its horizontal/vertical distance (x,y) from a reference point $(0,0)$, or by its overall distance from the origin and the angle the vector pointing to it makes with a reference axis (usually the x axis)



$P=(x,y)=(r,\theta)$

Converting: $r=\sqrt{x^2+y^2}$ $\quad \theta=\tan^{-1}(y/x)$

$x=r\cos\theta$ $\quad y=r\sin\theta$

Let's use this and our knowledge of partial derivatives to convert

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

to polar coordinates

Due to our conversions above, we can regard x and y as functions of r and θ, so that $u(x,y)=u(x(r,\theta),y(r,\theta))$. We can use this and the chain rule to compute the 2D-wave equation in polar coordinates.

We have:

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cdot(\cos\theta)+\frac{\partial u}{\partial y}\cdot(\sin\theta) \qquad (1)$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}&=\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\\&=\cos\theta\cdot\left(\frac{(\partial u/\partial x)}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{(\partial u/\partial x)}{\partial y}\cdot\frac{\partial y}{\partial r}\right)+\sin\theta\cdot\left(\frac{(\partial u/\partial y)}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{(\partial u/\partial y)}{\partial y}\cdot\frac{\partial y}{\partial r}\right)\\&=\cos\theta\cdot\left(\frac{\partial^2 u}{\partial x^2}\cdot\cos\theta+\frac{\partial^2 u}{\partial y\partial x}\cdot\sin\theta\right)+\sin\theta\cdot\left(\frac{\partial^2 u}{\partial x\partial y}\cdot\cos\theta+\frac{\partial^2 u}{\partial y^2}\cdot\sin\theta\right)\\&=\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2} \qquad (2)\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial\theta^2}=r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)-r\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right) \qquad (3)$$

We can notice that

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}&=\cos^2\theta\frac{\partial^2 u}{\partial x^2}+\cancel{2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}+\sin^2\theta\frac{\partial^2 u}{\partial x^2}-\cancel{2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\\&\quad-\frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)\\&=(\sin^2\theta+\cos^2\theta)\frac{\partial^2 u}{\partial x^2}+(\sin^2\theta+\cos^2\theta)\frac{\partial^2 u}{\partial y^2}-\frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)\\&=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{r}(\underbrace{\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}}_{\partial u/\partial r})\end{aligned}$$

So we have $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}$

Using the Frobenius method, we can show that the solution of the previous diff. equations is

$$R(r) = AJ_n(\tfrac{\omega}{c}r) + BY_n(\tfrac{\omega}{c}r), \text{ where}$$

$$J_n(r) = \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m+n}}{2^{2m+n} m!(n+m)!} \text{ is the Bessel function of the first kind}$$

and

$$Y_n(r) = \frac{2}{\pi} J_n(r) \int \frac{dr}{r J_n(r)^2} \text{ is Neumann's Bessel function of the second kind}$$

However, at the center of the drumhead ($r=0$), Y_n has a singularity that goes to ∞, so the only physical solution is

$$R(r) = AJ_n(\tfrac{\omega}{c}r)$$

Therefore, our total solution is of the form

$$u(r,\theta,t) = AJ_n(\tfrac{\omega}{c}r)\sin(\omega t+\phi)\sin(n\theta+\psi)$$

If the radius of the drum is $r=a$, our boundary condition is that $u=0$ when $r=a$ for all values of t and θ (since the drum head is restrained at its edges).

Therefore, we must have that $J_n(\frac{\omega}{c}a)=0$, which constrains the value of ω to certain discrete values, just like in the string and bar cases.

The first few zeroes of J_n are given below

zero #	J_0	J_1	J_2	J_3	J_4
1	2.40483	3.83171	5.13562	6.38016	7.58834
2	5.52008	7.01559	8.41724	9.76102	11.06471
3	8.65373	10.17347	11.61984	13.01520	14.37254

To choose a vibrational mode, choose a nonnegative integer n (the order of the Bessel function) and a zero of J_n. If the kth zero of J_n is denoted by $j_{n,k}$, the corresponding vibrational mode obeys

$$\frac{\omega_{n,k}}{c}a = j_{n,k}$$

so

$$f_{n,k} = \frac{c}{2\pi a} j_{n,k} = \frac{1}{2\pi a}\sqrt{\frac{T}{\rho}}\, j_{n,k}$$

Or, in relation to the fundamental frequency (i.e. the frequency of $j_{0,1}=2.40483$),

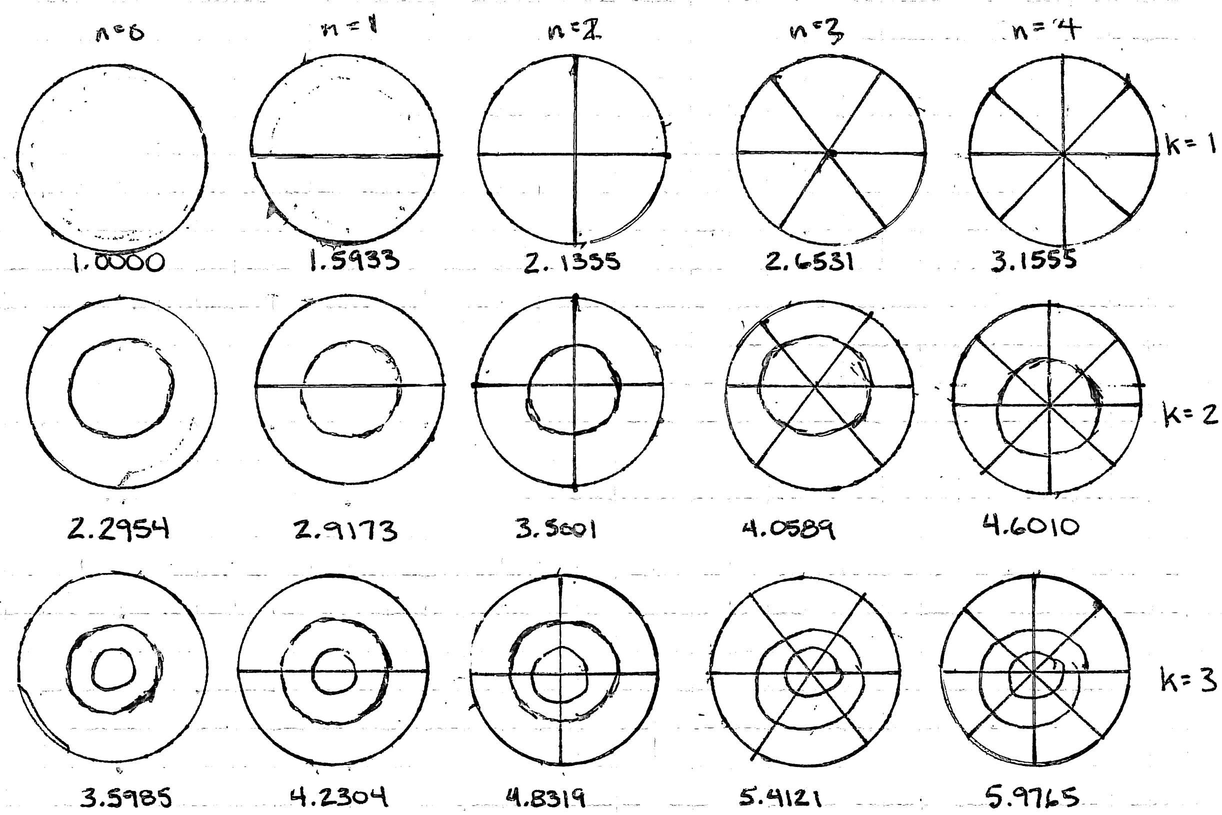
$$f_{n,k} = \underbrace{\frac{2.40483}{2\pi a}\sqrt{\frac{T}{\rho}}}_{f_{0,1}} \frac{j_{n,k}}{j_{0,1}}$$

We notice that for increasing k (higher zeroes) the frequency of vibration increases, meaning more nodal rings can fit on the drumhead (More circles of height 0 for all t). Increasing n affects the $\sin(n\theta+\psi)$ part of u, making the zeroes of this term more frequent. Since this is a function of θ, it creates nodal "diameters" on the drumhead with more for higher values of n. The first few modes [(n,k) pairs] and the associated $\frac{j_{n,k}}{j_{0,1}}$ are shown. (multiplying $\frac{j_{n,k}}{j_{0,1}}$ by the fundamental frequency $f_{0,1}$ gives the frequency $f_{n,k}$ of the (n,k)th mode). These pictures only show the stationary points ("nodes") of the vibrations, i.e. those points on the drum head that are always at height 0.

Since ω depends on n and k, so will A, ϕ, and θ for each standing wave, so we get

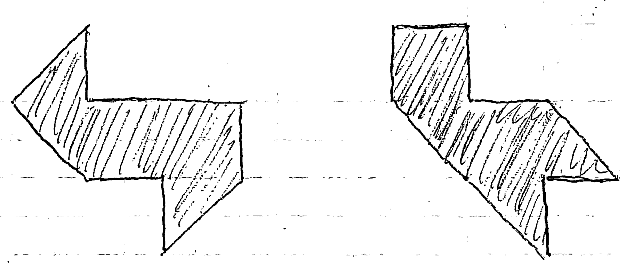
$$u_{n,k}(r,\theta,t) = A_{n,k} J_n(\tfrac{\omega_{n,k}}{c}r)\sin(\omega_{n,k}t+\phi_{n,k})\sin(n\theta+\psi_{n,k})$$

The overall wave behavior is given by a sum over all modes.



Can you hear the shape of a drum?

From the vibrating string to the drumhead, we've been deriving information about the frequencies of an apparatus given its geometry. In 1966, Mark Kac asked the converse. Given the set of frequencies, can we determine the shape of the drum uniquely?
To answer this, we can ask whether we can find two geometries that have the same spectrum.
It turns out we can, but they're not typical geometries. In the case of squares and rectangles, given their spectrums, they will always correspond to distinct geometries. However, the following two distinct geometries do have the same spectrums:



So barring other physical issues, these two drums would sound exactly the same.