Energy of a Vibrating String

We can find the total energy in a vibrating string by considering the energy present in each normal mode and adding all of these energies up, since any vibration of the string can be considered a superposition of these modes.
Similar to a harmonic oscillator, an elastic string alternates between having all kinetic energy and all potential energy when vibrating in one of its normal modes. Further, the total energy of the normal mode at any time is the sum of the kinetic energy and potential energy of the mode.
Therefore, we can find the total energy of a string vibrating in any one of its normal modes by finding its maximum kinetic (or potential) energy:

If we consider the energy of a small segment of the string vibrating in one of its normal modes, dE_n, we have that

$$dE_n = \frac{1}{2}\underbrace{(\mu dx)}_{m}\underbrace{\left(\frac{\partial y_n}{\partial t}_{max}\right)^2}_{v}$$

Recalling that the displacement at position x and time t of the nth normal mode is given by

$$\begin{aligned} y_n(x,t) &= (A_n \sin(\omega_n t) + B_n \cos(\omega_n t))\sin(k_n x) \\ &= \sqrt{A_n^2+B_n^2}\cos(\omega_n t + \tan^{-1}(-\tfrac{B_n}{A_n}))\sin(k_n x) \\ &= C_n \cos(\omega_n t + \phi_n)\sin(k_n x), \end{aligned}$$

where $C_n = \sqrt{A_n^2+B_n^2}$, and $\phi_n = \tan^{-1}(\frac{-B_n}{A_n})$.

Therefore,

$$\frac{\partial y_n}{\partial t} = -C_n\omega_n \sin(\omega_n t+\phi_n)\sin(k_n x)$$

So $$\frac{\partial y_n}{\partial t}_{max} = C_n\omega_n \sin(k_n x)$$

and thus

$$dE_n = \frac{1}{2}\mu C_n^2\omega_n^2 \sin^2(k_n x)\,dx.$$

Integrating this over the whole string gives the total energy of the nth normal mode

$$\begin{aligned} E_n &= \frac{1}{2}\mu C_n^2\omega_n^2 \int_0^L \sin^2(k_n x)\,dx \\ &= \frac{1}{2}\mu C_n^2\omega_n^2\left(\frac{L}{2}\right) \\ &= \frac{\omega_n^2\mu L}{4}C_n^2 \\ &= \frac{m}{4}C_n^2\omega_n^2, \end{aligned}$$

where m is the total mass of the string.

Therefore, the total energy in the string at any time is given by

$$E = \sum_n E_n = \sum_{n=1}^{\infty}\frac{m}{4}C_n^2\omega_n^2$$

As an example, we can find the total energy in a string plucked at $\frac{L}{2}$. In this case, we found that

$$A_n = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right), \quad B_n = 0$$

So

$$C_n^2 = A_n^2 + B_n^2 = \frac{64h^2}{n^4\pi^4}\sin^2\left(\frac{n\pi}{2}\right)$$

And

$$E_n = \frac{m}{4}C_n^2\omega_n^2 = \frac{m}{4}\left(\frac{64h^2}{n^4\pi^4}\sin^2\left(\frac{n\pi}{2}\right)\right)\omega_n^2 = m\left(\frac{4hc}{\pi L}\right)^2\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}$$

And thus, the total energy in a string plucked halfway along its length is

$$E = \sum_{n=1}^{\infty} m\left(\frac{4hc}{\pi L}\right)^2\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}$$

$$= m\left(\frac{4hc}{\pi L}\right)^2\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}$$

$$= m\left(\frac{4hc}{\pi L}\right)^2\sum_{k=0}\frac{1}{(2k+1)^2}$$

$$= m\left(\frac{4hc}{\pi L}\right)^2\cdot\frac{\pi^2}{8}$$

$$= 2m\left(\frac{hc}{L}\right)^2$$

$$= 2h^2\cdot\frac{T}{L}$$

Typically, the series won't be this nice, but we can follow the same procedure to find the total energy as a sum of the normal mode energies.