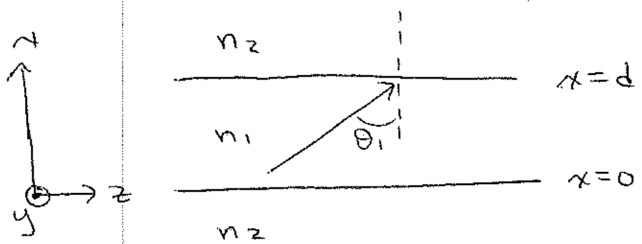
MODE PROFILE

LET'S DO THE SYMMETRIC, TE CASE FOR THE GUIDED MODES.



THE z-COMPONENT OF THE $\vec{k}$-VECTOR IS GIVEN BY

$$k_{z_1} = n_1 k_0 \sin\theta_1 \qquad \text{IN CORE}$$

AND

$$k_{z_2} = n_2 k_0 \sin\theta_2 \qquad \text{IN CLADS}$$

BY SNELL'S LOW $\quad n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad$ SO BOTH z-COMPONENTS ARE EQUAL. WE LET THE EFFECTIVE INDEX $N = n_1 \sin\theta_1$ SO

$$k_z = N k_0 = \beta \qquad (1)$$

THE x-COMPONENT OF $\vec{k}$ IS GIVEN BY

$$k_{x_1} = n_1 k_0 \cos\theta_1 \qquad \text{IN CORE}$$

AND

$$k_{x_2} = n_2 k_0 \cos\theta_2 \qquad \text{IN CLAD}$$

IN THE CORE, WE CAN WRITE

$$k_{x_1} = k_0\sqrt{n_1^2 - n_1^2 \sin^2\theta_1} = k_0\sqrt{n_1^2 - N^2} \equiv K_1 \qquad (2)$$

IN THE CLADS, WE CAN WRITE

$$\begin{aligned} k_{x_2} &= k_0\sqrt{n_2^2 - n_2^2\sin^2\theta_2} \\ &= k_0\sqrt{n_2^2 - n_1^2\sin^2\theta_1} && \text{USING SNELL'S LOW} \\ &= i\,k_0\sqrt{n_1^2\sin^2\theta_1 - n_2^2} && \text{SINCE WE HAVE TIR} \\ &= i\,k_0\sqrt{N^2 - n_2^2} \end{aligned}$$

$$k_{x_2} = i\,\gamma_2 \qquad \text{WHERE} \quad \gamma_2 = \sqrt{N^2 - n_2^2}\,k_0 \qquad (3)$$

WE KNOW THE WAVES ARE OF THE FORM

CORE $E_y = E_{01}\, e^{\pm i k_{x_1} x}\, e^{i(k_{z_1} z - \omega t)}$ ← NOTE: THE $\pm$ MODELS WAVES "BOUNCING" UP & DOWN IN x-DIRECTION

CLADS $E_y = E_{02}\, e^{\pm i k_{x_2} x}\, e^{i(k_{z_2} z - \omega t)}$ ← NOTE: THE + SIGN FOR THE TOP CLAD, THE − SIGN FOR THE BOTTOM CLAD

THESE BECOME

CORE $E_y = E_{01}\, e^{\pm i K_1 x}\, e^{i(\beta z - \omega t)}$

CLADS $E_y = E_{02}\, e^{\pm \gamma_2 x}\, e^{i(\beta z - \omega t)}$

NOTE WE GET AN EXPONENTIAL DECAY IN THE CLADS FOR $\mp x$.

NOW WE LOOK AT OUR x-AXIS VALUES AND WRITE THE SPECIFIC FORMS:

TOP CLAD	$x > d$	$E_y = E_{02}\, e^{-\gamma_2 (x-d)}\, e^{i(\beta z - \omega t)}$ (4A)
CORE	$0 < x < d$	$E_y = \left(A e^{i K_1 x} + B e^{-i K_1 x}\right) e^{i(\beta z - \omega t)}$ (4B)
BOTTOM CLAD	$x < 0$	$E_y = E_{02}\, e^{\gamma_2 x}\, e^{i(\beta z - \omega t)}$ (4C)

WE HAVE WRITTEN THE UPGOING WAVE & THE DOWNWARD WAVE WITH SEPARATE AMPLITUDES A & B. THIS IS REQUIRED FOR GENERALITY (AND CORRECTNESS!)

ARE WE SURE THESE FORMS ARE CORRECT. WELL, THEY DO SATISFY THE WAVE EQUATION. WATCH!

THE WAVE EQUATION IS OF THE FORM

$$\nabla^2 \vec{E} = \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} \qquad (5)$$

OR

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_y = \frac{1}{v^2} \frac{\partial^2 E_y}{\partial t^2}$$

NOW $\frac{\partial^2 E_y}{\partial y^2} = 0$ AND $v = \frac{\omega}{K}$ SO WE GET

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = \frac{k^2}{\omega^2} \frac{\partial^2 E_y}{\partial t^2} \qquad (6)$$

IN THE CORE & CLADS, $\dfrac{\partial^2 E_y}{\partial z^2} = -\beta^2 E_y$

AND $\dfrac{\partial^2 E_y}{\partial t^2} = -\omega^2 E_y.$

IN THE CORE, $\dfrac{\partial^2 E_y}{\partial x^2} = -K_1^2 E_y$

IN THE CLADS, $\dfrac{\partial^2 E_y}{\partial x^2} = +\gamma_2^2 E_y$

SO WE HAVE (6) BECOMING

- IN CORE: $-K_1^2 - \beta^2 = -k^2 \rightarrow$

$$K_1^2 = k^2 - \beta^2 = n_1^2 k_0^2 - N^2 k_0^2$$

$$K_1 = k_0 \sqrt{n_1^2 - N^2} \qquad (7)$$

THIS AGREES WITH (2).

- IN CLADS: $+\gamma_2^2 - \beta^2 = -k^2 \rightarrow$

$$\cancel{\gamma_2^2 = k^2 - \beta^2 = n_2^2 k_0^2 - N^2}$$

$$\gamma_2^2 = \beta^2 - k^2 = N^2 k_0^2 - n_2 k_0^2$$

$$\gamma_2 = k_0 \sqrt{N^2 - n_2^2} \qquad (8)$$

THIS AGREES WITH (3).

OKAY, SO WE HAVE VALID MATHEMATICAL SOLUTIONS. NOW LET'S APPLY THE PHYSICAL BOUNDARY CONDITIONS THAT E_y AND $\dfrac{\partial E_y}{\partial x}$ BE CONTINUOUS ACROSS THE INTERFACES.

① E_y CONTINUOUS AT :

- $x = d \rightarrow \quad Ae^{iK_1 d} + Be^{-iK_1 d} = E_{02}$ (9A)
- $x = 0 \rightarrow \quad A + B = E_{02}$ (9B)

② $\frac{\partial E_y}{\partial x}$ CONTINUOUS AT :

- $x = d \rightarrow \quad iK_1 \left(Ae^{iK_1 d} - Be^{-iK_1 d}\right) = -\gamma_2 E_{02}$ (10A)
- $x = 0 \rightarrow \quad iK_1 (A - B) = \gamma_2 E_{02}$ (10B)

SUB (9A) INTO (10A) AND (9B) INTO (10B) TO GET

$$iK_1 \left(Ae^{iK_1 d} - Be^{-iK_1 d}\right) = -\gamma_2 \left(Ae^{iK_1 d} + Be^{-iK_1 d}\right) \quad (11A)$$

$$iK_1 (A - B) = \gamma_2 (A + B) \quad (11B)$$

REARRANGING GIVES :

$$(K_1 - i\gamma_2) Ae^{iK_1 d} - (K_1 + i\gamma_2) Be^{-iK_1 d} = 0 \quad (12A)$$

$$(K_1 + i\gamma_2) A - (K_1 - i\gamma_2) B = 0 \quad (12B)$$

LET $C = K_1 + i\gamma_2$. (13)

USING THE RULES OF COMPLEX ALGEBRA, WE CAN WRITE THIS AS

$$C = |C| e^{i\alpha} = |C| \cos\alpha + i |C| \sin\alpha$$

NOTE THAT $K_1 = |C| \cos\alpha$ AND $\gamma_2 = |C| \sin\alpha$ SO THAT

$$\tan\alpha = \frac{\gamma_2}{K_1} \quad (14)$$

LO AND BEHOLD, THIS α IS THE SAME α THAT APPEARS IN THE PHASE SHIFT FOR TIR WAVES! YOU CAN SUBSTITUTE IN γ_2 & K_1 AND CHECK THIS IF YOU WANT.

USING $C = K_1 + i\gamma_2 = |C|e^{i\alpha}$, (12A) & (12B) BECOME

$$C^* A e^{iK_1 d} - C B e^{-iK_1 d} = 0 \qquad (15A)$$

$$CA - C^* B = 0 \qquad (15B)$$

C^* = COMPLEX CONJUGATE OF C

SOLVING (15B) FOR B GIVES $B = \frac{C}{C^*} A = \frac{|C|e^{i\alpha}}{|C|e^{-i\alpha}} A = e^{i2\alpha} A$ (16)

SUBBING THIS B INTO (15A) GIVES

$$\left(C^* e^{iK_1 d} - C e^{i2\alpha} e^{-iK_1 d}\right) A = 0 \qquad (17)$$

FOR NON-ZERO A, WE MUST HAVE

$$C^* e^{iK_1 d} - C e^{i2\alpha} e^{-iK_1 d} = 0$$

$$1 - \left(\frac{C}{C^*}\right) e^{i2\alpha} e^{-i2K_1 d} = 0$$

$$1 - \left(e^{i2\alpha}\right) e^{i2\alpha} e^{-i2K_1 d} = 0$$

$$e^{i(4\alpha - 2K_1 d)} = 1$$

SO $4\alpha - 2K_1 d = 2m'\pi$ m' = INTEGER

OR $\boxed{K_1 d - 2\alpha = m\pi}$ m = INTEGER (18)

THIS IS THE TE DISPERSION EQUATION !!

$$n_1 k_0 \cos\theta_1 - 2\alpha = m\pi$$

WE CAN NOW FIND A RELATIONSHIP BETWEEN THE AMPLITUDES OF THE WAVES IN THE CORE & CLADS.

WE FOUND IN (16) THAT $B = e^{i2\alpha} A$. SUBBING THIS INTO (9B) GIVES

$$E_{oz} = A + e^{i2\alpha} A$$

SO

$$A = \frac{E_{oz}}{1+e^{i2\alpha}} = \frac{e^{-i\alpha}}{e^{-i\alpha}} \cdot \frac{E_{oz}}{1+e^{i2\alpha}} = \frac{e^{-i\alpha} E_{oz}}{e^{-i\alpha}+e^{i\alpha}}$$

$$A = \frac{e^{-i\alpha}}{2\cos\alpha} E_{oz} \qquad (19A)$$

THEN

$$B = e^{i2\alpha} A = \frac{e^{i\alpha}}{2\cos\alpha} E_{oz} \qquad (19B)$$

SUBBING A & B INTO THE WAVE FORM OF 4(B) GIVES FOR THE CORE

$$E_y = \frac{E_{oz}}{2\cos\alpha}\left(e^{-i\alpha} e^{iK_1x} + e^{i\alpha} e^{-iK_1x}\right) e^{i(\beta z-\omega t)}$$

$$= \frac{E_{oz}}{2\cos\alpha}\underbrace{\left(e^{i(K_1x-\alpha)} + e^{-i(K_1x-\alpha)}\right)}_{2\cos(K_1x-\alpha)} e^{i(\beta z-\omega t)}$$

$$= \left(\frac{E_{oz}}{\cos\alpha}\right) \cos(K_1x-\alpha)\, e^{i(\beta z-\omega t)}$$

$$E_y = E_{o_1} \cos(K_1x-\alpha)\, e^{i(\beta z-\omega t)}$$

WHERE

$$E_{o_1} = \frac{E_{oz}}{\cos\alpha} \qquad (20)$$

WE CAN REWRITE (20) RECALLING (14)

$$\tan\alpha = \frac{\gamma_2}{K_1}$$

SO

$$\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1 = \frac{\gamma_2^2}{K_1^2} \qquad \left(\begin{array}{c}\sin^2\alpha+\cos^2\alpha=1\\ \tan^2\alpha+1=\frac{1}{\cos^2\alpha}\end{array}\right)$$

OR $$\frac{1}{\cos^2\alpha} = 1 + \frac{\gamma_2^2}{K_1^2} = \frac{K_1^2+\gamma_2^2}{K_1^2}$$

THUS, (20) BECOMES UPON SQUARING

$$E_{o1}^2 = \left(\frac{K_1^2+\gamma_2^2}{K_1^2}\right) E_{o2}^2$$

OR, SOLVING FOR E_{o2},

$$E_{o2}^2 = \frac{K_1^2}{K_1^2+\gamma_2^2}\, E_{o1}^2$$

SUBSTITUTING (2) & (3) FOR γ_2 & K_1 GIVES

$$E_{o2}^2 = \frac{k_0^2\,(n_1^2-N^2)}{k_0^2\,(n_1^2-N^2)+k_0^2\,(N^2-n_2^2)}\; E_{o1}^2$$

OR

$$E_{o2} = \sqrt{\frac{n_1^2-N^2}{n_1^2-n_2^2}}\; E_{o1} \qquad (21)$$

<u>SUMMARIZING</u>: FOR ARBITRARY z, t, SAY $t=0$ & $z=0$

$$\begin{array}{ll} x>d & E_y = E_{o2}\, e^{-\gamma_2 (x-d)} \\ 0<x<d & E_y = E_{o1} \cos(K_1 x - \alpha) \\ x<0 & E_y = E_{o2}\, e^{\gamma_2 x} \end{array}$$

WHERE
$$K_1 = k_0\sqrt{n_1^2-N^2}$$
$$\gamma_2 = k_0\sqrt{N^2-n_2^2}$$
$$\tan\alpha = \frac{\gamma_2}{K_1}$$
$$E_{o2} = \sqrt{\frac{n_1^2-N^2}{n_1^2-n_2^2}}\; E_{o1}$$

SQUARING THESE E_y FUNCTIONS GIVES THE INTENSITY FUNCTIONS WHERE $I_{o1} = E_{o1}^2$, $I_{o2} = E_{o2}^2$ AND

$$I_{o2} = \left(\frac{n_1^2-N^2}{n_1^2-n_2^2}\right) I_{o1}$$

SINCE THE INTENSITY $I \propto |E_y|^2$, WE CAN WRITE

$$x > d \qquad I = I_{02}\, e^{-2\gamma_2 (x-d)}$$

$$0 < x < d \qquad I = I_{01} \cos^2(k_1 x - \alpha)$$

$$x < 0 \qquad I = I_{02}\, e^{2\gamma_2 x}$$

WHERE $I_{02} = \left(\dfrac{n_1^2 - N^2}{n_1^2 - n_2^2} \right) I_{01}$