and

$$\vec{\mathbf{B}} = -B\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$$

$$\vec{\mathbf{B}}_r = B_r\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} \tag{23-16}$$

$$\vec{\mathbf{B}}_t = -B_t\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega t)}$$

Requiring continuity of the components of the electric and magnetic fields that are parallel to the boundary gives, in this case,

$$-B + B_r = -B_t \tag{23-17}$$

$$E\cos\theta + E_r\cos\theta = E_t\cos\theta_t \tag{23-18}$$

Reflection and Transmission Coefficients

The magnetic field amplitudes of Eqs. (23-14) and (23-17) can be expressed in terms of the corresponding electric field amplitudes through the generic relation

$$E = vB = \left(\frac{c}{n}\right)B \tag{23-19}$$

Writing the index of refraction for incident and refracting media as n_1 and n_2, respectively, Eqs. (23-12), (23-14), (23-17), and (23-18) can be recast as follows:

$$\text{TE:}\begin{cases} E + E_r = E_t & (23\text{-}20)\\ n_1 E\cos\theta - n_1 E_r\cos\theta = n_2 E_t\cos\theta_t & (23\text{-}21)\end{cases}$$

$$\text{TM:}\begin{cases} -n_1 E + n_1 E_r = -n_2 E_t & (23\text{-}22)\\ E\cos\theta + E_r\cos\theta = E_t\cos\theta_t & (23\text{-}23)\end{cases}$$

Next, eliminating E_t from each pair of equations and solving for the *reflection coefficient* $r = E_r/E$,

$$r_{TE} = \frac{E_r}{E} = \frac{\cos\theta - n\cos\theta_t}{\cos\theta + n\cos\theta_t} \tag{23-24}$$

$$r_{TM} = \frac{E_r}{E} = \frac{-n\cos\theta + \cos\theta_t}{n\cos\theta + \cos\theta_t} \tag{23-25}$$

where we have introduced a *relative refractive index* $n \equiv n_2/n_1$. Note that we use subscripts to distinguish between the TE and TM cases. Finally, since n and θ_t are related to θ through Snell's law, $\sin\theta = n\sin\theta_t$, θ_t may be eliminated using

$$n\cos\theta_t = n\sqrt{1-\sin^2\theta_t} = \sqrt{n^2 - \sin^2\theta} \tag{23-26}$$

The results are then

$$\boxed{r_{TE} = \frac{E_r}{E} = \frac{\cos\theta - \sqrt{n^2-\sin^2\theta}}{\cos\theta + \sqrt{n^2-\sin^2\theta}}} \tag{23-27}$$

$$r_{TM} = \frac{E_r}{E} = \frac{-n^2 \cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2 \cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{23-28}$$

Returning to Eqs. (23-20) through (23-23), if E_r is eliminated instead of E_t, similar steps lead to the following equations describing the *transmission coefficient* $t = E_t/E$:

$$t_{TE} = \frac{E_t}{E} = \frac{2\cos\theta}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{23-29}$$

$$t_{TM} = \frac{E_t}{E} = \frac{2n\cos\theta}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{23-30}$$

Eqs. (23-29) and (23-30) can also be found more quickly by using Eqs. (23-20) and (23-22) written in the form

$$t_{TE} = 1 + r_{TE}$$

$$nt_{TM} = 1 - r_{TM}$$

into which the results expressed by Eqs. (23-27) and (23-28) can be conveniently substituted. Equations (23-27) through (23-30) are the *Fresnel equations*, giving reflection and transmission coefficients, the ratio of both reflected and transmitted $\vec{\mathbf{E}}$-field amplitudes to the incident $\vec{\mathbf{E}}$-field amplitude. Note that, for normal incidence, the reflection and transmission coefficients for the TE case are identical to those for the TM case. This is sensible since for normal incidence there is no distinction between the two cases.[1] In practice, measured reflection and transmission coefficients also depend on scattering losses from a nonplanar surface.

Example 23-1

Calculate the reflection and transmission coefficients for both TE and TM modes of light incident from air at 30° onto glass of index 1.60.

Solution

Using Eqs. (23-27) and (23-28),

$$r_{TE} = \frac{\cos(30°) - \sqrt{1.6^2 - \sin^2(30°)}}{\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}} = -0.2740$$

$$r_{TM} = \frac{-1.6^2\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}}{1.6^2\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}} = -0.1866$$

Using the relations below Eq. (23-30),

$$t_{TE} = 1 + r_{TE} = 1 - 0.2740 = 0.7260$$

$$t_{TM} = \frac{1 - r_{TM}}{n} = \frac{1 + 0.1866}{1.60} = 0.7416$$

[1]Some texts use a different convention, in which the positive direction of the reflected electric field for the TM case is opposite to that shown in Figure 23-2, leading to an expression for the reflection coefficient for the TM case that differs from ours by a factor of −1. Of course, both conventions lead to the same physical result since the extra factor of −1 simply reverses the direction of the reflected electric field.

23-2 EXTERNAL AND INTERNAL REFLECTIONS

When interpreting these equations, it is useful to distinguish between two physically different situations:

$$\text{external reflection:} \quad n_1 < n_2 \quad \text{or} \quad n = \frac{n_2}{n_1} > 1$$

$$\text{internal reflection:} \quad n_1 > n_2 \quad \text{or} \quad n = \frac{n_2}{n_1} < 1$$

Figure 23-3 is a plot of Eqs. (23-27) through (23-30) for the case of external reflection with $n = 1.50$. Notice that at both normal and grazing incidence—angles of 0° and 90°, respectively—TE and TM modes have reflection coefficients of the same magnitude and transmission coefficients of the same magnitude. Negative values of r for both the TE and TM modes indicate a phase change of the $\vec{\mathbf{E}}$- or $\vec{\mathbf{B}}$-field vectors on reflection and will be discussed presently. The fraction of power P in the incident wave that is reflected or transmitted, called the *reflectance* and the *transmittance*, respectively, depends on the ratio of the squares of the amplitudes.

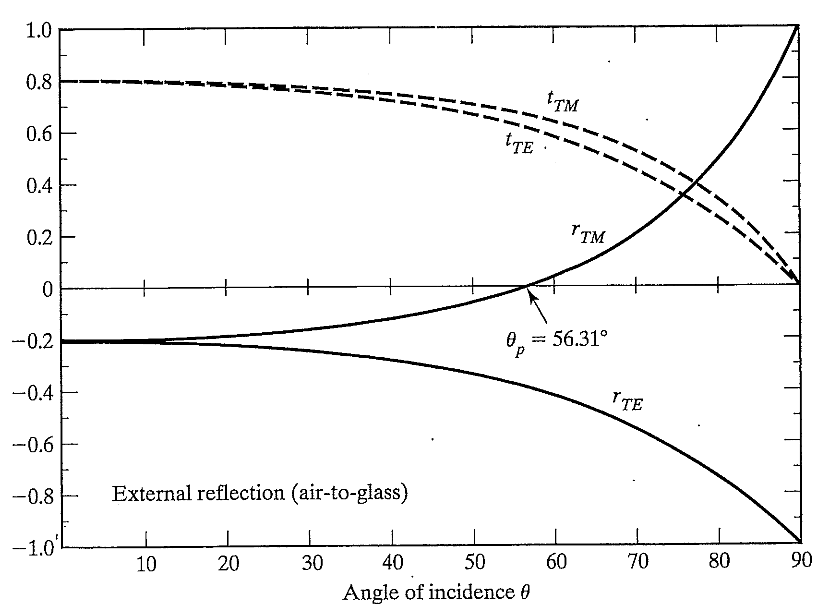
$$\text{reflectance} = R = \frac{P_r}{P_i} = r^2 = \left(\frac{E_r}{E}\right)^2 \tag{23-31}$$

$$\text{transmittance} = T = \frac{P_t}{P_i} = n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{23-32}$$

These expressions are justified later in this chapter.

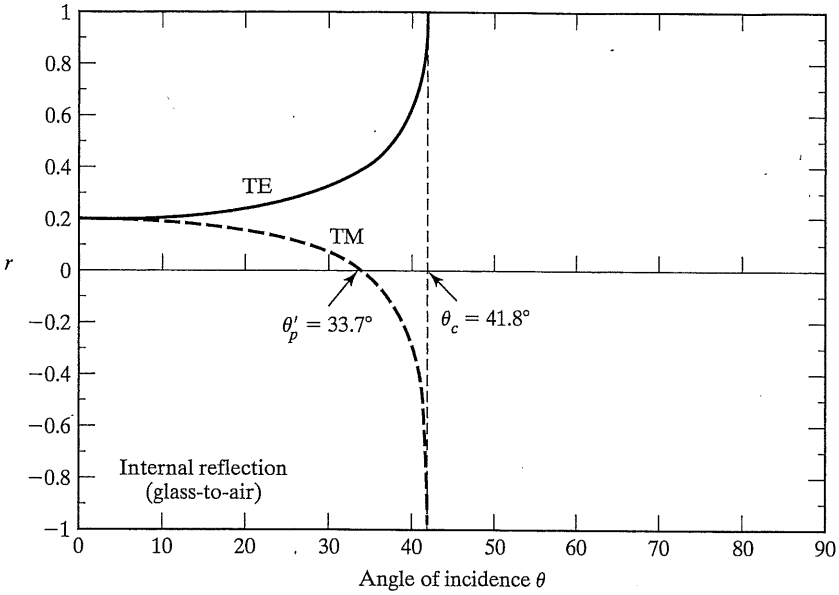
In Figure 23-4, reflectance is plotted as a function of the angle of incidence θ. The curve for the case of external reflection, TM mode, indicates that no wave energy is reflected when the angle of incidence is near 60°. The angle θ_p at which $R_{TM} = 0$ is known as *Brewster's angle* or the *polarizing* angle and takes the value,

$$\theta_p = \tan^{-1}(n) = \tan^{-1}(n_2/n_1)$$



PLOT 1

Figure 23-3 Reflection and transmission coefficients for the case of external reflection, with $n = n_2/n_1 = 1.50$.



PLOT 2

Figure 23-5 Reflection coefficient for the case of internal reflection with $n = n_1/n_2 = 1/1.50$.

23-3 PHASE CHANGES ON REFLECTION

The negative values of the reflection coefficient in Figures 23-3 and 23-5 indicate that $E_r = -|r|E$ in certain situations. Evidently, the electric field vector may reverse direction on reflection. Equivalently, in such cases there is a π-phase shift of E on reflection, as the following mathematical argument demonstrates:

$$E_r = -|r|E = e^{i\pi}|r|E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = |r|E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t+\pi)}$$

Thus in the case of external reflection, Figure 23-3, a π-phase shift of E occurs at any angle of incidence for the TE mode and for $\theta < \theta_p$ for the TM mode. When reflection is internal, Figure 23-5, we conclude that a π-phase shift occurs for the TM mode for $\theta_p' < \theta < \theta_c$. However, the situation in the region $\theta > \theta_c$, where r is complex, requires further investigation. When $\theta > \theta_c = \sin^{-1}(n)$, the radical in Eqs. (23-27) and (23-28) becomes imaginary, and the equations may be written in the form

$$r_{TE} = \frac{\cos\theta - i\sqrt{\sin^2\theta - n^2}}{\cos\theta + i\sqrt{\sin^2\theta - n^2}} \tag{23-34}$$

$$r_{TM} = \frac{-n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}} \tag{23-35}$$

The reflection coefficients can be written in polar form as $r = |r|e^{i\phi}$ and we shall refer to ϕ as the *phase shift on reflection.* In Eq. (23-34), the reflection coefficient takes the form $r_{TE} = (a - ib)/(a + ib)$. Since the real and imaginary parts of the numerator and denominator are the same, except for a sign, the magnitudes of the numerator and denominator are equal, and r_{TE} has unit amplitude. The phase of r_{TE} may be investigated by expressing Eq. (23-34) in complex polar form, as

$$r_{TE} = \frac{e^{-i\alpha}}{e^{i\alpha}} = e^{-i(2\alpha)}$$

where $\tan\alpha = \sqrt{\sin^2\theta - n^2}/\cos\theta$. So, for the TE case, the phase shift on reflection is $\phi_{TE} = -2\alpha$. A similar analysis (see problem 23-6) can be used to

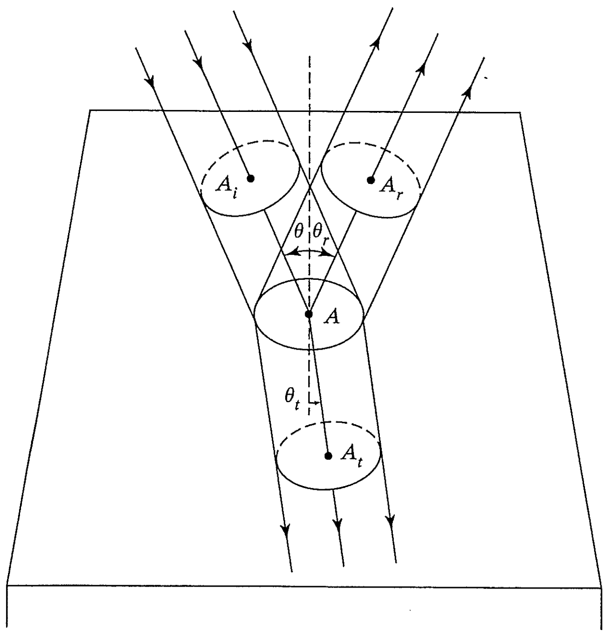


Figure 23-9 Comparison of cross sections of incident, reflected, and transmitted beams.

The quantity $(v_t\varepsilon_t/v_i\varepsilon_i)$ is just a complicated way of expressing the relative refractive index n, which we can show as follows:

$$\frac{v_t\varepsilon_t}{v_i\varepsilon_i} = \frac{v_t\, v_i^2\, \mu_i}{v_i\, v_t^2\, \mu_t} = \frac{v_i}{v_t} = n \tag{23-45}$$

In arriving at this result we have used

$$\mu_i = \mu_t = \mu_0$$

for nonmagnetic materials and the relation

$$v^2 = \frac{1}{\mu\varepsilon}$$

for the velocity of a plane electromagnetic wave. Incorporating Eq. (23-45) in Eq. (23-44),

$$E_{0i}^2 = E_{0r}^2 + n\left(\frac{\cos\theta_t}{\cos\theta}\right)E_{0t}^2 \tag{23-46}$$

Dividing the equation by the left member, it becomes

$$1 = r^2 + n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{23-47}$$

where the reflection and transmission coefficients r and t have been introduced. Now the quantity r^2 is just the reflectance R:

$$R = \frac{P_r}{P_i} = \frac{I_r}{I_i} = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

Comparing Eq. (23-47) with Eq. (23-42), it follows that the transmittance T is expressed by the relation

$$T = n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{23-48}$$

PLOT 3

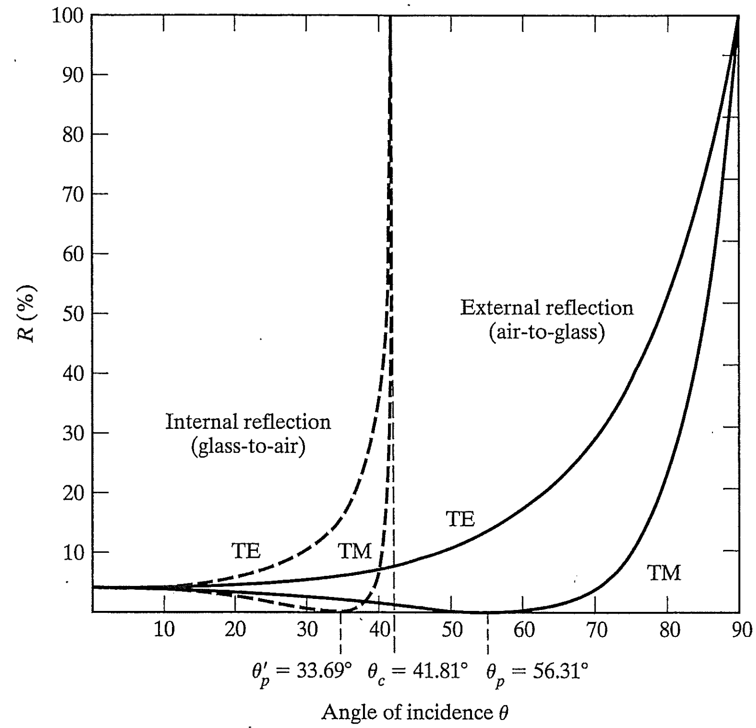


Figure 23-4 Reflectance for both external and internal reflection when $n_1 = 1$ and $n_2 = 1.50$.

This condition is also evident in the vanishing of r_{TM} in Figure 23-3 and the vanishing of the numerator of Eq. (23-28). See problem 23-1. For the case $n = 1.50$ used in Figures 23-3 and 23-4, $\theta_p = 56.31°$. R_{TE} does not go to zero under this condition, so reflected light contains only the TE mode and is linearly polarized, with $R_{TE} = 15\%$. At normal incidence ($\theta = 0°$), for both TE and TM modes, Eqs. (23-24) and (23-25) simplify to give

$$R = r^2 = \left(\frac{1 - n}{1 + n}\right)^2 \tag{23-33}$$

Equation (23-33) gives a reflectance of 4% from an air/glass interface with $n = 1.5$. Keep in mind, however, that n is a function of wavelength. As the angle of incidence increases to grazing incidence ($\theta = 90°$), both R_{TE} and R_{TM} become unity, although R_{TM} remains quite small until Brewster's angle has been exceeded.

The reflection coefficient for the case of internal reflection is shown in Figure 23-5 with $n = 1/1.50$, as when light encounters a glass/air interface from the glass side. Evidence of phase changes and of a polarizing, or Brewster's, angle may also be seen here. For the case of internal reflection we give Brewster's angle the symbol θ'_p. Examination of Figures 23-4 and 23-5 shows that, for the case of internal reflection, both $R_{TE} = r_{TE}^2$ and $R_{TM} = r_{TM}^2$ reach values of unity before the angle of incidence θ reaches 90°. This is the phenomenon of *total internal reflection*, which occurs at the critical angle $\theta_c = \sin^{-1}(n) = \sin^{-1}(n_2/n_1)$. For the example of glass ($n = 1/1.5$) used in Figure 23-5, $\theta'_p = 33.7°$ and $\theta_c = 41.8°$. When $\sin\theta_c > n$, the radical $\sqrt{n^2 - \sin^2\theta}$ is negative and both r_{TE} and r_{TM} are complex. Their magnitudes, however, are easily shown to be unity in this range, giving total reflection for $\theta > \theta_c$.

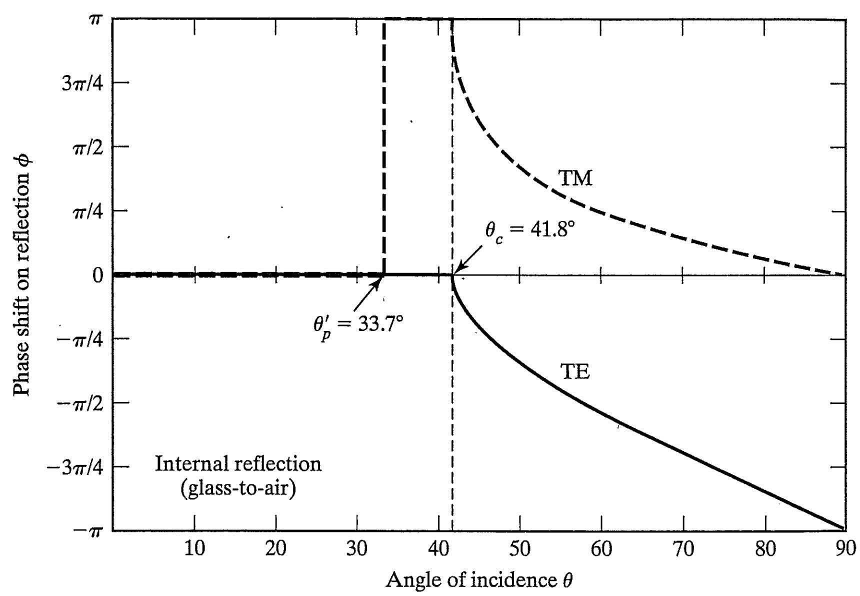
show that r_{TM}, like r_{TE}, has unit magnitude when the angle of incidence exceeds the critical angle and enables one to find the phase shift on total internal reflection ϕ_{TM} for the TM case. The phase shifts on *total internal reflection* for the two cases have the form,

$$\tan\left(\frac{\phi_{TE}}{2}\right) = -\frac{\sqrt{\sin^2\theta - n^2}}{\cos\theta} \tag{23-36}$$

$$\tan\left(\frac{\phi_{TM} - \pi}{2}\right) = -\frac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta} \tag{23-37}$$

Clearly, the phase shift on reflection, for total internal reflection, may take on values other than 0 and π, depending on the angle of incidence. The phase shift ϕ, as determined from Eqs. (23-36) and (23-37), is plotted in Figure 23-6.

PLOT 4



ure 23-6 Phase shift ϕ on reflection of electric field for internally reflected 's, with $n = n_1/n_2 = 1/1.5$.

It happens that the relative phase shift $\phi_{TE} - \phi_{TM}$ is about $-3\pi/4$ at an angle of incidence near 53°. Two consecutive internal reflections thus produce a relative phase shift of $2(-3\pi/4) = -3\pi/2$ (equivalently, $+\pi/2$) between the perpendicular components of the $\vec{\mathbf{E}}$-field. Recall that circularly polarized light consists of equal amplitude components with phases that differ by $\pm\pi/2$. Thus linearly polarized incident light with equal TM and TE components, after two internal reflections at 53°, will be transformed into circularly polarized light. This technique is utilized in the *Fresnel rhomb* (Figure 23-7).

Summarizing these results for the case of internal reflection,

$$\phi_{TM} = \begin{cases} 0, & \theta < \theta'_p \\ \pi, & \theta'_p < \theta < \theta_c \\ -2\arctan\left(\dfrac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta}\right) + \pi, & \theta > \theta_c \end{cases} \tag{23-38}$$

$$\phi_{TE} = \begin{cases} 0, & \theta < \theta_c \\ -2\arctan\left(\dfrac{\sqrt{\sin^2\theta - n^2}}{\cos\theta}\right), & \theta > \theta_c \end{cases} \tag{23-39}$$

Phase shifts for both TM and TE modes and for both internal and external reflection are summarized in Figure 23-8.

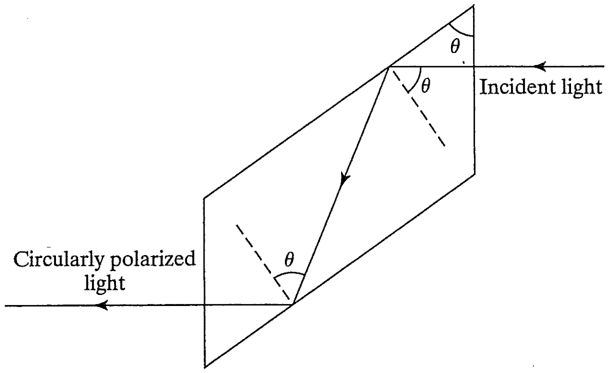
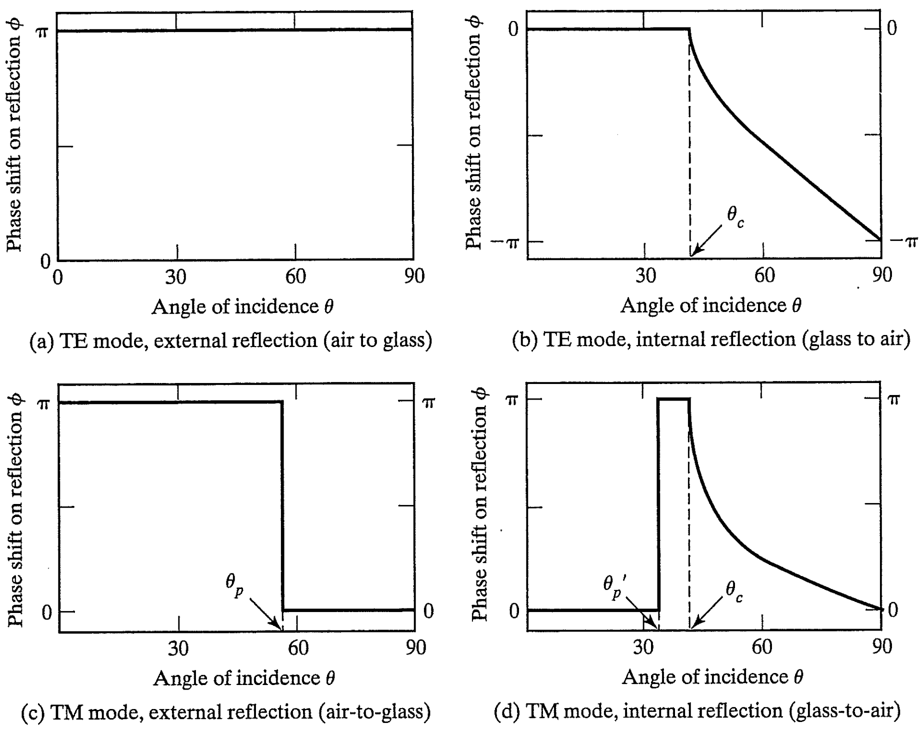


Figure 23-7 The Fresnel rhomb. With the incident light polarized at 45° to the plane of incidence, two internal reflections produce equal-amplitude TE and TM amplitudes with a relative phase of $\pi/2$, or circularly polarized light. For $n = 1.50$, the angle should be $\theta = 53°$. The device is effective over a wide range of wavelengths.



PLOT 5

Figure 23-8 Phase changes on reflection ϕ between incident and reflected rays versus angle of incidence. Discontinuities occur at $\theta_c = 41.8°$, $\theta_p = 56.3°$, and $\theta'_p = 33.7°$ for refractive indices of $n_1 = 1$ and $n_2 = 1.50$.

Example 23-2

What is the phase shift of the TM and TE rays reflected both externally and internally for the situation discussed in Example 23-1?

Solution

For this interface,

$$\theta_c = \sin^{-1}\left(\frac{1}{1.6}\right) = 38.7°$$

$$\theta_p = \tan^{-1}(1.6) = 58.0°$$

$$\theta'_p = \tan^{-1}\left(\frac{1}{1.6}\right) = 32.0°$$

Since the angle of incidence of 30° is less than either θ'_p or θ_c, Eqs. (23-38) and (23-39) or Figure 23-8 require that for internal reflection, $\phi_{TM} = 0$ and $\phi_{TE} = 0$, while Figure 23-8 shows that for external reflection, $\phi_{TM} = \pi$ and $\phi_{TE} = \pi$.