Motion in Magnetic Fields

This section has some very useful results, namely we can find a way to determine the electron density of a material and have a means to determine that some semiconductor materials carry current with positive charges!

1. Equations of motion

We start with Newton's Second Law:

$$\Sigma\mathbf{F} \;=\; \mathbf{F}_{el} + \mathbf{F}_{mag} + \mathbf{F}_{scat} \;=\; m\,d\mathbf{v}/dt \quad .$$

Now we already know what the electrical and magnetic forces look like in terms of the electric and magnetic fields. But what does the scattering force look like? Recall that when we considered only a constant electric force, we introduced the idea of scattering via collisions with phonons, and we obtained the following expression involving the average time between collisions, τ:

$$\delta\mathbf{k} \;=\; (-e\mathbf{E}/\hbar)\,\tau \; .$$

We note that the electric force, $-e\mathbf{E}$, was balanced by the scattering force such that the average velocity was constant. Hence we can infer that the scattering force "balanced" the electric force to cause the constant average velocity that carried the current. Hence we will identify the scattering force as that which is equal to $-e\mathbf{E}$:

$$\mathbf{F}_{scat} \;=\; -\hbar\,\delta\mathbf{k}/\tau \; .$$

Further, we recall that $\hbar\mathbf{k} = \mathbf{p} = m\mathbf{v}$, so that $\mathbf{F}_{scat} = -m\delta\mathbf{v}/\tau$, and the $\delta\mathbf{v}$ is the resulting $\mathbf{v}$, Newton's Second Law becomes:

$$-e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - m\mathbf{v}/\tau \;=\; m(d\mathbf{v}/dt) \; .$$

2. Special case of magnetic field being constant and in the z direction

If we **assume** that $\mathbf{B} = B\mathbf{z}$, (where $\mathbf{z}$ is a unit vector in the z direction) then when we write out Newton's Second law in component form, we get the following three equations:

$$m\,dv_x/dt \;=\; -eE_x - eBv_y - mv_x/\tau \quad ,$$

$$m\,dv_y/dt \;=\; -eE_y + eBv_x - mv_y/\tau \quad ,$$

$$m\,dv_z/dt \;=\; -eE_z - mv_z/\tau \quad .$$

Further, if we **assume** a steady state motion, then the $d/dt = 0$, so:

$$v_x \;=\; (-e\tau/m)\,E_x + (-eB\tau/m)\,v_y \quad ,$$

$$v_y \;=\; (-e\tau/m)\,E_y - (-eB\tau/m)\,v_x \quad ,$$

$$v_z \;=\; (-e\tau/m)\,E_z \quad .$$

Note from the above equation that (eB/m) has the units of inverse time (since τ has the units of time, and v_x and v_y have the same units of distance/time). Recall that the magnetic force is perpendicular to the velocity. This gives us circular motion:

$$qvB \;=\; ma \;= mv^2/r \quad , \quad \text{or} \quad r \;=\; mv/qB \quad ,$$

and $v = \omega r$ for circular motion, (we will call ω_c the **cyclotron frequency**)

so that: $r = m(\omega_c r)/qB$, or $\boldsymbol{\omega_c = qB/m}$.

3. The Hall Effect

Consider a rod with a length in the x direction and a diameter perpendicular to the x direction. We apply a constant electric field in the x direction ($E_x = V/L_x$ where V is the voltage applied) and a constant magnetic field, B, in the z direction.

What should we expect for this situation? First, there should be a flow of electrons opposite the direction of the applied electric field due to the electric force on the electrons. Hence v_x should not be zero. Second, since we have a moving charge going through a magnetic field, we would expect that charge to try to turn in a circle. But due to the finite thickness of the wire, the electron may not be able to make the complete circle. Hence, the electrons will tend to turn toward one end of the wire, creating a net accumulation of electrons on that side of the wire, and hence creating a net electric field in the wire directed across the wire!

Now let's set up the equations. We also assume the steady state case (thus the d/dt is zero). Note that the velocities in the y and z directions must on average be zero (since the electrons have nowhere to go in those directions). Thus the three component equations become:

$$m\,dv_x/dt = -eE_x - eBv_y - mv_x/\tau \quad \rightarrow \quad v_x = (-e\tau/m)E_x - 0\,,$$

$$m\,dv_y/dt = -eE_y + eBv_x - mv_y/\tau \quad \rightarrow \quad 0 = (-e\tau/m)E_y + (eB\tau/m)v_x\,,$$

$$m\,dv_z/dt = -eE_z - mv_z/\tau \quad \rightarrow \quad 0 = (-e\tau/m)E_z\,.$$

The third equation simply says that there should be no electric field in the z direction (that is, in the direction of the applied magnetic field). The first equation says that there will be a v_x due to the E_x (as we anticipated). The second equation says that an electric field will be created in the wire in the direction perpendicular to both the applied E (x) and B (z) directions, i.e., in the y direction:

$$E_y = [(eB\tau/m)v_x]/(e\tau/m) = Bv_x\,.$$

And since the existence of this electric field implies a voltage difference, we will call this voltage difference the **HALL voltage**, V_H:

$$V_H = \int E_y\,dy = E_y L_y = Bv_xL_y\,.$$

Further, we define a **Hall coefficient, R_H**:

$$R_H \equiv E_y/j_xB\,.$$

The reason this coefficient is important is due to the following: since $E_y = Bv_x$, and $j_x = I_x/A_{yz} = -nev_x$ where n is the electron density, the coefficient becomes simply:

$$R_H = [Bv_x]/[(-nev_x)B] = 1/[-ne]\,.$$

Thus by measuring the dimensions of the rod [used to get L_y and A_{yz}], the hall voltage [thus getting $E_y=V_y/L_y$], the regular current [thus getting $j_x=I_x/A_{yz}$], and the magnetic field **we can calculate the density of electrons** in the material: n.

$$R_H \equiv E_y/j_xB = 1/[-ne] \quad \text{or} \quad n = j_xB/-eE_y\,.$$

In addition, for semi-conductors we sometimes find that the sign of the hall coefficient is reversed - which indicates that the current is actually carried by positive charges!

Also by using a known material, we can use the Hall effect to **determine magnetic field strengths** by measuring Hall voltages!

Here are some experimental values for R_H (at room Temperature) versus some calculated values of 1/[-ne] (for comparison) and the assumed carriers/atom:

Metal	R_H in 10^{-11} m^3/Coul	1/[-ne] in 10^{-11} m^3/Coul	assumed carriers/atom
Cu	-5.4	- 7.3	1 electron
Ag	-9.0	- 10.7	1 electron
Al at 4K	+10.2	+10.2	1 hole*
Al at room T	-3.5	-3.5	3 electrons

*We'll see what a "hole" is in Part 4.

The first value for Al at 4K above is from table 4, page 151 in Kittel (6^{th} edition).
The second value for Al above is from the web:
http://www.mhhe.com/engcs/electrical/kasap2/graphics/ColorDiagramsPDF/Tables_PDF/table-ch2-2nd.pdf

Example: Cu has an atomic weight of 63 grams/mole and a density of 8.9 gm/cm^3, so there should be (63 grams/mole) / (8.9 gm/cm^3) = 7.08 cm^3/mole = 7.08 x 10^{-6} m^3 / 6.02 x 10^{23} atoms = 1.18 x 10^{-29} m^3/atom and so 1/[-ne] = -1.18 x 10^{-29} m^3 / 1.6 x 10^{-19} Coul = 7.35 x 10^{-11} m^3/Coul.

Note: in measuring R_H, we measure the current density ($j_x = I_x/A_{yz} = -nev_x$) which is related to the speed of the electrons ($v_x = (-e\tau/m)E_x$ from above), which is related to the mass of the electrons, whereas using the 1/[-ne] formula we don't use the mass of the electrons; but as we saw before, we may have to use an "effective mass" for the electrons. So it should not be surprising that the values in the above table are close but not exactly the same.