ELECTRICAL CONDUCTIVITY (D.C.)

[Note: we will consider D.C. conductivity here. It is your problem (actually two: ***homework problems 25 and 26***) to derive a similar expression for the A.C. electrical conductivity.]

1. Wave-Particle Duality

How do we consider electrical conductivity if the electrons travel as waves instead of particles? Another way of stating this question is: how do we relate the speed of the charged electrons to the wave (which has a frequency and wavelength)? Answer: recall the DeBroglie relation

$$\lambda = h/p \text{ , or alternately: } 2\pi/\lambda = k = p/\hbar \text{ , or } m\mathbf{v} = \mathbf{p} = \hbar\mathbf{k} \text{ .}$$

2. Starting point: Newton's Second Law

To really do this problem, we would need to start with Schrodinger's Equation and put in PE = qV with the voltage a function of position and time. Instead, we can derive an electrical conductivity a little easier by using Newton's Second Law and the fact that the electrical force depends on the electric (and magnetic) field(s):

$$\Sigma\mathbf{F} = \mathbf{F}_{el} + \mathbf{F}_{mag} = -e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = m\, d\mathbf{v}/dt = \hbar\, d\mathbf{k}/dt \text{ .}$$

3. Including the Pauli Exclusion Principle: the Fermi sphere

It is important to see how charge can move with electrons that obey the Pauli Exclusion Principle. If there is no net force and zero temperature, then the electrons will fill up the possible energy states up to the Fermi energy level. This means that most of the electrons will in fact be moving! But the states will also be filled up in a symmetrical way so that the NET velocity, and hence **NET** momentum and wavevector, will be **zero**. We can think of the states being specified by their quantum numbers, or equivalently by their k_x, k_y, and k_z values. (Recall that the quantum numbers and wavevectors are related: $k_x = n_x 2\pi/L_x$.) If we also envision a k-space (that is, axes in terms of k_x, k_y, and k_z instead of the regular space axes of x, y, and z), then the filled k-states will form a sphere (called the **Fermi sphere**) **in k space**. This sphere will have a radius of $k_F = [k_x^2 + k_y^2 + k_z^2]^{½}$.

If we now allow non-zero temperature (but still zero net force), then some of the electrons will jump to higher energy states which are outside of the Fermi sphere and hence leave empty states (holes) inside the Fermi sphere. But the distribution should remain essentially symmetric so that the **NET** velocity (and momentum and wavevector) will **remain zero**.

If we now include an external force (such as that due to the electric or magnetic field), the electrons will jump to different energy states but in such a way that the **NET** velocity, momentum and wavevector will **no longer remain zero**. Electrons near the surface of the Fermi sphere in the direction of the force will be able to move outside the Fermi sphere in the direction of the force. Those further inside the Fermi sphere will be able to move into the holes left by those first electrons moving outside the sphere. Finally, those electrons on the far side of the Fermi sphere will see holes toward the interior of the Fermi sphere and will be able to move toward those holes. Thus the entire Fermi sphere should be displaced by the force, and hence the net velocity, momentum and wavevector should be changed from zero to a value in the direction of the applied force.

4. $\mathbf{E}$ = constant, $\mathbf{B}$ = zero

For the case of a constant electric field (and no magnetic field), Newton's Second Law gives us:

$$-e\mathbf{E} = m\, d\mathbf{v}dt = \hbar\, d\mathbf{k}/dt$$ (here e is considered a positive value, so we need to put in the – sign into the equation)

which is a simple differential equation with the obvious solution:

$$\mathbf{k}(t) = \mathbf{k}(0) - (e\mathbf{E}/\hbar)\, t \quad ,$$

which says that the wavevector (and hence the velocity, momentum and hence the charge flow [or current]) will continue to increase with time. But this does not agree with experiment since a voltage difference (which creates an electric field) will result in a **constant** current rather than a continually increasing current. Therefore, we must be leaving something important out of Newton's Second Law, particularly some force.

5. Collisions and collision times

The one force that might "slow" the electrons down so that they do not continue to accelerate is that due to collisions.

If one electron collides with another electron, we must have conservation of momentum in the collision, so the net momentum (and hence velocity and wavevector) would not change, and so the net current would not change. Hence this is not the mechanism to slow the current down so that it is constant and does not continually increase.

If one electron collides with the lattice, or more particularly with a phonon, then the electron could lose momentum to the lattice. When this happens, the electron should on average lose all sense of its initial direction (because it might glance off with little change or it might collide head on which would actually reverse its direction - or anything inbetween). Thus any electron might accelerate during the time between collisions, but then after a collision it would (on average) be back to its initial velocity. This then can slow the electrons down so that on average the current remains constant. The net change in wavevector, $\delta\mathbf{k}$, during the time between collisions, τ, would be:

$$\delta\mathbf{k} = (-e\mathbf{E}/\hbar)\,\tau = m\,\delta\mathbf{v}/\hbar \quad , \quad \text{and} \quad \delta\mathbf{v} = (-e\mathbf{E}/m)\,\tau$$

where **τ is the average time between collisions** (of electrons with phonons - or with lattice defects).

Now since velocity is related to wavevector ($m\mathbf{v} = \mathbf{p} = \hbar\mathbf{k}$), and since current is related to electron velocity, we have for the current density:

$$\mathbf{j} = \text{current/area} = [(\text{charge/sec})/\text{area}]\,[d_\perp/d_\perp] \quad ,$$

or $$\mathbf{j} = [\text{charge/vol}]\,[d_\perp/\text{time}] = n(-e)\delta\mathbf{v} \quad ,$$ where n is the valence electron density

or $$\mathbf{j} = n(-e)(-e\mathbf{E}\tau/m) = (ne^2\tau/m)\,\mathbf{E} \quad ,$$

or, $$\mathbf{j} = \sigma\mathbf{E} \; , \qquad \text{where} \qquad \boldsymbol{\sigma = ne^2\tau/m}$$

and σ is called the (D.C.) electrical conductivity. [Here are three example values: Al: 3.65×10^5 /(Ohm-cm), Cu: 5.88×10^5 /(Ohm-cm), Fe: 1.02×10^5 /(Ohm-cm)] If we define resistivity as: $\boldsymbol{\rho = 1/\sigma}$, then we can write:

$$\mathbf{E} = \rho\mathbf{j} \, .$$

Further, recall that voltage and current are related to field and current density:

$$V = \int \mathbf{E} \bullet d\boldsymbol{\ell} = E\ell \quad \text{(for constant fields)},$$

where V is the voltage difference between the two ends of the conductor that are separated by a length, ℓ,

$$I = \int \mathbf{j} \bullet d\mathbf{A} = j\,A \quad \text{(for uniform current densities)} \, ,$$

where A is the cross-sectional area of the conductor, so that:

$$V = E\,\ell = (\rho\, j)\,\ell = \rho\,(I/A)\,\ell = (\rho\,\ell/A)\,I = R\,I$$

or: $$V = I\,R$$ which we recognize as Ohm's Law, where $\boldsymbol{R = \rho\,\ell/A}$.

6. Mean free path

Since only those electrons near the Fermi surface are going to jump to different available energy levels (those near the forward surface jump to higher, those near the back surface are able to fill in the "holes" left by the other jumping electrons), **the mean free path, *l***, should approximately be:

$$l \approx v_F \tau .$$

7. Temperature dependence of σ

$$\sigma = ne^2\tau/m$$

Note that the dependence of conductivity, σ, on temperature comes from the dependence of n and τ on temperature. For metals, the electron density ("free" or valence electron density), n, is essentially constant; and so σ should basically depend on the temperature dependence of τ which should depend on the average number of phonons. Basically, the higher the temperature, the more phonons and so the lower τ and so the lower the conductivity and so the higher the resistivity and so the higher the resistance.

For semiconductors, we will see that the number of "free" or valence electrons depends strongly on temperature and this turns out to be the major determining factor for conductivity for semiconductors.

However, as the temperature decreases, the resistivity does not go to zero due to the presence of impurities and defects. These defects cause some residual resistivity and depend on the specimen (rather than the material as the phonon density does).

*You should now be able to do **homework problems #25, 26, and 27**.*