GROUP VELOCITY

Group of waves with possibly different phase velocities:

1. Pure sinusoidal wave: $Y(x,t) = A\cos(kx-\omega t+\theta_o) = A\cos(\theta)$

where $\theta = \theta(x,t)$ = phase angle = $kx-\omega t+\theta_o$

where $k = 2\pi$ radians$/\lambda$ (kx is then an angle)

where $\omega = 2\pi$ radians$/T$ (ωt is then an angle)

and θ_o is the initial phase angle (at x=0 when t=0)

[$\theta_o = 0^o$ for cosine wave; $\theta_o = -90^o$ for sine wave.]

For crest of wave (phase angle of crest is constant at 90° = ½π rad):

$$\theta_{crest} = \tfrac{1}{2}\pi = kx_{crest} - \omega t_{crest} + \theta_o, \quad \text{or}$$

$$x_{crest} = (\tfrac{1}{2}\pi + \omega t_{crest} - \theta_o)/k \ .$$

For speed of crest of wave: $v_{phase} = v_{crest} = dx_{crest}/dt_{crest}$, so

$$v_{phase} = d([\tfrac{1}{2}\pi + \omega t - \theta_o]/k)/dt = \omega/k$$ (here $t = t_{crest}$

for short); recall that $k=2\pi/\lambda$ and $\omega=2\pi/T$ so that:

$v_{phase} = (2\pi/T)/(2\pi/\lambda) = \lambda/T$, and since $f = (1/T)$

$$\mathbf{v_{phase} = \omega/k = \lambda f} \ .$$

2. Group of waves: A group of sine waves will add together to form some pattern that also repeats (this is the Fourier Series in reverse).

$$Y_{group}(x,t) = A(x)\cos(Kx - \omega_g t) \quad \text{where}$$

$A(x)$ is the shape of the group,

$K = 2\pi/\lambda_g$ where λ_g is the distance over which the pattern for the group repeats, and

$\omega_g = 2\pi/T_g$ where T_g is the time over which the pattern for the group repeats.

At t=0 sec, $Y_{group}(x,0) = A(x)\cos(Kx)$ where $A(x) = {}_n\Sigma\, a_n \cos(k_n x)$

(here $A(x)$ is expressed as a Fourier Series) , so

$$Y_{group}(x,0) = {}_n\Sigma\, a_n \cos(k_n x)\cos(Kx) \ .$$

We can now use two trig identities

$[\cos(\theta\pm\varphi) = \cos\theta\cos\varphi -/+ \sin\theta\sin\varphi]$

to get $\cos\theta\cos\varphi = \tfrac{1}{2}[\cos(\theta+\varphi) + \cos(\theta-\varphi)]$,

and with $\theta=k_n x$ and $\varphi=Kx$, we get

$$Y_{group}(x,0) = {}_n\Sigma\, \tfrac{1}{2} a_n \{\cos[(k_n+K)x] + \cos[(k_n-K)x]\} \ ,$$

and since $\cos(-\theta) = \cos(+\theta)$, we can write:

$$Y_{group}(x,0) = {}_n\Sigma\, \tfrac{1}{2} a_n \{\cos[(K+k_n)x] + \cos[(K-k_n)x]\} \ .$$

Now put in the time dependence such that wherever we had a Kx, we put in an additional $-\omega t$: $(Kx) \rightarrow (Kx-\omega t)$:

$$Y_{group}(x,t) = {}_n\Sigma \tfrac{1}{2} a_n \{ \cos[(K+k_n)x-\omega_+ t] + \cos[(K-k_n)x-\omega_- t] \}$$

where we use $\omega_\pm$ to indicate that ω depends on $k=(K\pm k_n)$.

[Recall that $v_{phase} = \omega/k$, and v_{phase} may not be constant but may depend on (vary with) ω.] Since ω is a function of k [$\omega(k) = v_{phase}k$], we can expand $\omega(k)$ in a Taylor Series about k=K: normally the "carrier wave" is a high frequency wave with small wavelength and so large K, and the "signal" is a set of lower frequency waves with larger wavelength values and so smaller k values.

$$\omega(K\pm k_n) = \omega(K) \pm (d\omega/dk)_K k_n + \text{higher order terms which we neglect} ;$$

now let's **let $v_g \equiv (d\omega/dk)_K$** so that $\omega_\pm = \omega(K\pm k_n) \approx \omega(K) \pm v_g k_n$, so

$$Y_{group}(x,t) = {}_n\Sigma \tfrac{1}{2} a_n \{\cos[(K+k_n)x - (\omega+v_g k_n)t] + \cos[(K-k_n)x - (\omega-v_g k_n)t] \}$$

or re-grouping terms:

$$Y_{group}(x,t) = {}_n\Sigma \tfrac{1}{2} a_n \{ \cos[(Kx-\omega t)+k_n(x-v_g t)] + \cos[(Kx-\omega t)-k_n(x-v_g t)] \} .$$

We can again use our trig identity:

$$\cos(\theta+\varphi) + \cos(\theta-\varphi) = 2 \cos\theta \cos\varphi ,$$

where $\theta = (Kx-\omega t)$ and $\varphi = k_n(x-v_g t)$, to get:

$$Y_{group} = {}_n\Sigma a_n \cos(Kx-\omega t) \cos[k_n(x-v_g t)] ;$$

but here the $\cos(Kx-\omega t)$ can come out of the summation, so

$$Y_{group} = \{ {}_n\Sigma a_n \cos[k_n(x-v_g t)\} \cos(Kx-\omega t) = \mathbf{A(x-v_g t)\cos(Kx-\omega t)},$$

where we identify the function $A(x-v_g t)$ as the original Fourier series with x replaced by $(x-v_g t)$; that is, the shape moves through space with a speed of v_g, hence the name **group velocity**.

3. Review:

$$\mathbf{v_{phase} = \omega/k = \lambda f}$$

(good for any pure sine wave [or cosine wave] of wavelength λ and frequency f [or wavevector k and angular speed ω]) ;

$$\mathbf{v_{group} = d\omega/dk} .$$

4. Special case: If v_{phase} = constant, then $\omega = v_{phase}k$, and so

$$v_{group} = d\omega/dk = d[v_{phase}k]/dk = v_{phase} .$$