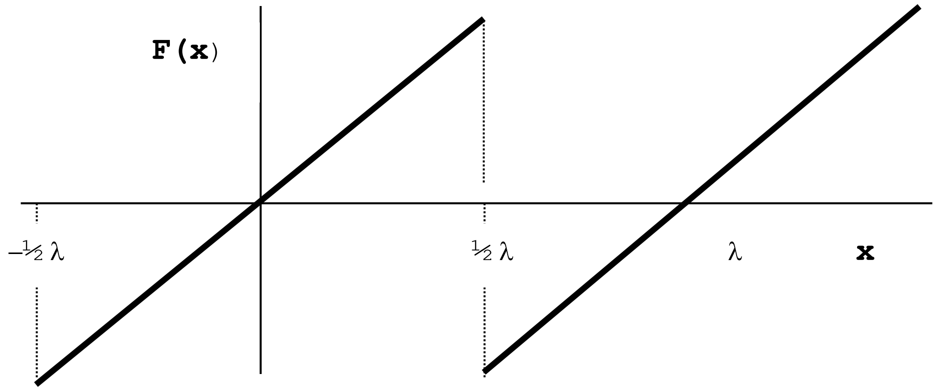
FOURIER SERIES: an example - the SAWTOOTH WAVE

1. Define the repeating function:



F(x) = Cx
for $\frac{1}{2}\lambda < x < \frac{1}{2}\lambda$

2. The general form for the Fourier series:

$$f(\theta) = \tfrac{1}{2} A_o + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta) \quad , \quad \text{where}$$

$$A_m = (1/\pi) \int_{\theta_o}^{\theta_o+2\pi} f(\theta) \cos(m\theta)\, d\theta, \quad B_m = (1/\pi) \int_{\theta_o}^{\theta_o+2\pi} f(\theta) \sin(m\theta)\, d\theta \ .$$

3. Convert from θ to x:

In our case, x repeats over a distance λ (just as θ repeats over an angle of 2π). Therefore, we make the substitution $\theta = 2\pi x/\lambda$
(you should see that as x goes from 0 to λ, θ goes from 0 to 2π):

$$f(x) = \tfrac{1}{2} A_o + \sum_{n=1}^{\infty} A_n \cos(n2\pi x/\lambda) + \sum_{n=1}^{\infty} B_n \sin(n2\pi x/\lambda) \quad , \quad \text{where}$$

$$A_m = (1/\pi) \int_{-\pi}^{+\pi} f(x) \cos(m2\pi x/\lambda)\, d(2\pi x/\lambda) \quad \text{and}$$

$$B_m = (1/\pi) \int_{-\pi}^{+\pi} f(x) \sin(m2\pi x/\lambda)\, d(2\pi x/\lambda) \ ;$$

or

$$A_m = (1/\pi)(2\pi/\lambda) \int_{-\frac{1}{2}\lambda}^{+\frac{1}{2}\lambda} f(x) \cos(m2\pi x/\lambda)\, dx \quad \text{and}$$

$$B_m = (1/\pi)(2\pi/\lambda) \int_{-\frac{1}{2}\lambda}^{+\frac{1}{2}\lambda} f(x) \sin(m2\pi x/\lambda)\, dx \ ;$$

or substituting in $f(x) = Cx$:

$$A_m = (2/\lambda) \int_{-\frac{1}{2}\lambda}^{+\frac{1}{2}\lambda} C\, x \cos(m2\pi x/\lambda)\, dx \quad \text{and}$$

$$B_m = (2/\lambda) \int_{-\frac{1}{2}\lambda}^{+\frac{1}{2}\lambda} C\, x \sin(m2\pi x/\lambda)\, dx \ .$$

4. Evaluate A_m and B_m :

Since cosine is an even function [$\cos(-\theta) = \cos(\theta)$] and $f(x) = Cx$ is an odd function [$f(-x) = -f(x)$], the integral about an interval symmetric about the origin will be zero. Hence **A_m = 0 for all m.**
[You can also integrate by parts to get this result.]

We can integrate $\int x \sin(k_m x)\, dx$ by parts (here $k_m = m2\pi/\lambda$) so that

$$B_m = (-1)^{m+1} \lambda C/m\pi \ .$$

5. Put A_m and B_m back into the Fourier series expression to get:

$$f(x) = (\lambda C/\pi) \sin(2\pi x/\lambda) + (-\lambda C/2\pi) \sin(4\pi x/\lambda) + (\lambda C/3\pi) \sin(6\pi x/\lambda) + \ldots$$

6. To see how well the series works, plot f(x) using the approximation of keeping only several different numbers of terms:

let $\lambda = 2$ meters, let $C = 3$ units/meter
(where units indicate the units of f itself)

plot using one term: (pink)
$f(x) = \lambda C/\pi \sin(2\pi x/\lambda) = (6/\pi)$ units $\sin([\pi \text{ rad/m}]\, x)$

plot using three terms (blue)
(see bottom of previous page & stop after the third term)

plot using the actual function: $f(x) = C\,x$ (dark)
(to see how close the above approximations come).