

FOURIER SERIES

for functions that are periodic in either space or time or both

Since a lattice has atoms in a periodic structure, Fourier Series will be a very useful tool!

(On this page, vectors will be displayed in **bold blue (A)**, unit vectors (unit magnitude used to indicate direction only) will be displayed in **bold purple (x)**, and scalars will be displayed in normal black (A_x).

1. The **idea** of Fourier Series is that of **breaking a vector into components**:

$$\mathbf{A} = A_x \mathbf{x} + A_y \mathbf{y} + A_z \mathbf{z} = \sum A_i \mathbf{x}_i \quad \text{where } \mathbf{x}_1 = \mathbf{x}, \mathbf{x}_2 = \mathbf{y}, \mathbf{x}_3 = \mathbf{z}$$

$$\text{where } A_x = \mathbf{A} \cdot \mathbf{x}, \quad A_y = \mathbf{A} \cdot \mathbf{y}, \quad A_z = \mathbf{A} \cdot \mathbf{z}, \quad \text{or } A_i = \mathbf{A} \cdot \mathbf{x}_i$$

$$\text{since } \mathbf{x} \cdot \mathbf{x} = 1, \text{ and } \mathbf{x} \cdot \mathbf{y} = 0, \quad \mathbf{x}_i \cdot \mathbf{x}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2. The **basis** of Fourier Series are the results:

$$\int_0^{2\pi} \sin(n\theta) \sin(m\theta) d\theta = \begin{cases} \pi & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_0^{2\pi} \cos(n\theta) \cos(m\theta) d\theta = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_0^{2\pi} \sin(n\theta) \cos(m\theta) d\theta = 0 \quad \text{for all } n, m \neq 0$$

(**Homework problem 7** asks you to show this via a diagram. An example is shown on the last page.)

3. The **method** of Fourier Series is:

$\mathbf{f}(\theta) = \sum A_n \cos(n\theta) + \sum B_n \sin(n\theta)$ where $f(\theta) = f(\theta+2\pi)$, i.e., f is a periodic function

$$\text{where } A_0 = (1/2\pi) \int_0^{2\pi} f(\theta) \cos(0) d\theta$$

$$A_n = (1/\pi) \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta \quad \text{similar to } A_i = \mathbf{A} \cdot \mathbf{x}_i$$

$$B_n = (1/\pi) \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta$$

Note: If $f(\theta)$ is an even function, i.e. $f(\theta) = f(-\theta)$, then $B_n = 0$ since the sine function is odd. If $f(\theta)$ is an odd function, i.e. $f(\theta) = -f(-\theta)$, then $A_n = 0$ since the cosine function is even.

4. The use of Fourier Series is for breaking up waves (whether in space or in time [or both]) into sine and cosine functions.

a) For functions that oscillate in time, θ must be replaced by ωt (where ω is the angular frequency in radians/sec), $d\theta$ by ωdt [recall that $\omega = 2\pi/T$ (where T is the period of oscillation)], and the limits of the integral by t_0 to t_0+T .

b) For functions that oscillate in space, θ must be replaced by kx (where k is the wavevector in radians/meter), $d\theta$ by $k dx$ [recall that $k = 2\pi/\lambda$ (where λ is the wavelength)], and the limits of the integral by x_0 to $x_0+\lambda$.

5. A Fourier Series can also be expressed in terms of exponentials due to the following identity (provable by Taylor Series Expansion):

$$e^{in\theta} = \cos(n\theta) + i \sin(n\theta) \quad , \quad \text{and} \quad e^{-in\theta} = \cos(n\theta) - i \sin(n\theta)$$

where n is considered positive. Therefore, we can write:

$$\cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \quad , \quad \text{and} \quad \sin(n\theta) = (1/i) \frac{1}{2}(e^{in\theta} - e^{-in\theta}) .$$

Therefore, we can write the Fourier Series as: (recall $1/i = i/i^2 = -i$)

$$\begin{aligned} f(\theta) &= \sum A_n \cos(n\theta) \quad + \quad \sum B_n \sin(n\theta) \\ &= \sum A_n \left\{ \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \right\} + \sum i B_n \left\{ -\frac{1}{2}(e^{in\theta} - e^{-in\theta}) \right\} \\ &= \sum \frac{1}{2}(A_n - iB_n) e^{in\theta} + \sum \frac{1}{2}(A_n + iB_n) e^{-in\theta} \quad (\text{where } n \text{ is always positive}). \end{aligned}$$

If we now let $C_{n>0} = \frac{1}{2}(A_n - iB_n)$, and let $C_{n<0} = \frac{1}{2}(A_n + iB_n)$

Note: $C_{n<0}$ is the complex conjugate of $C_{n>0}$.

(where n is +), and $C_0 = \frac{1}{2} A_0$, then we can write:

$$f(\theta) = \sum C_n e^{in\theta} \quad \text{for } n \text{ both } + \text{ and } - \text{ and } 0, \text{ where}$$

for $n>0$: $C_n = \frac{1}{2}(A_n - iB_n) = (1/2\pi) \int_{\theta_0}^{\theta_0+2\pi} f(\theta) \{ \cos(n\theta) - i \sin(n\theta) \} d\theta$,

or $C_n = (1/2\pi) \int_{\theta_0}^{\theta_0+2\pi} f(\theta) e^{-in\theta} d\theta$ (for $n>0$) , and

$$C_{-n} = \frac{1}{2}(A_n + iB_n) = (1/2\pi) \int_{\theta_0}^{\theta_0+2\pi} f(\theta) \{ \cos(n\theta) + i \sin(n\theta) \} d\theta,$$

or $C_{-n} = (1/2\pi) \int_{\theta_0}^{\theta_0+2\pi} f(\theta) e^{in\theta} d\theta$ ($n>0$).

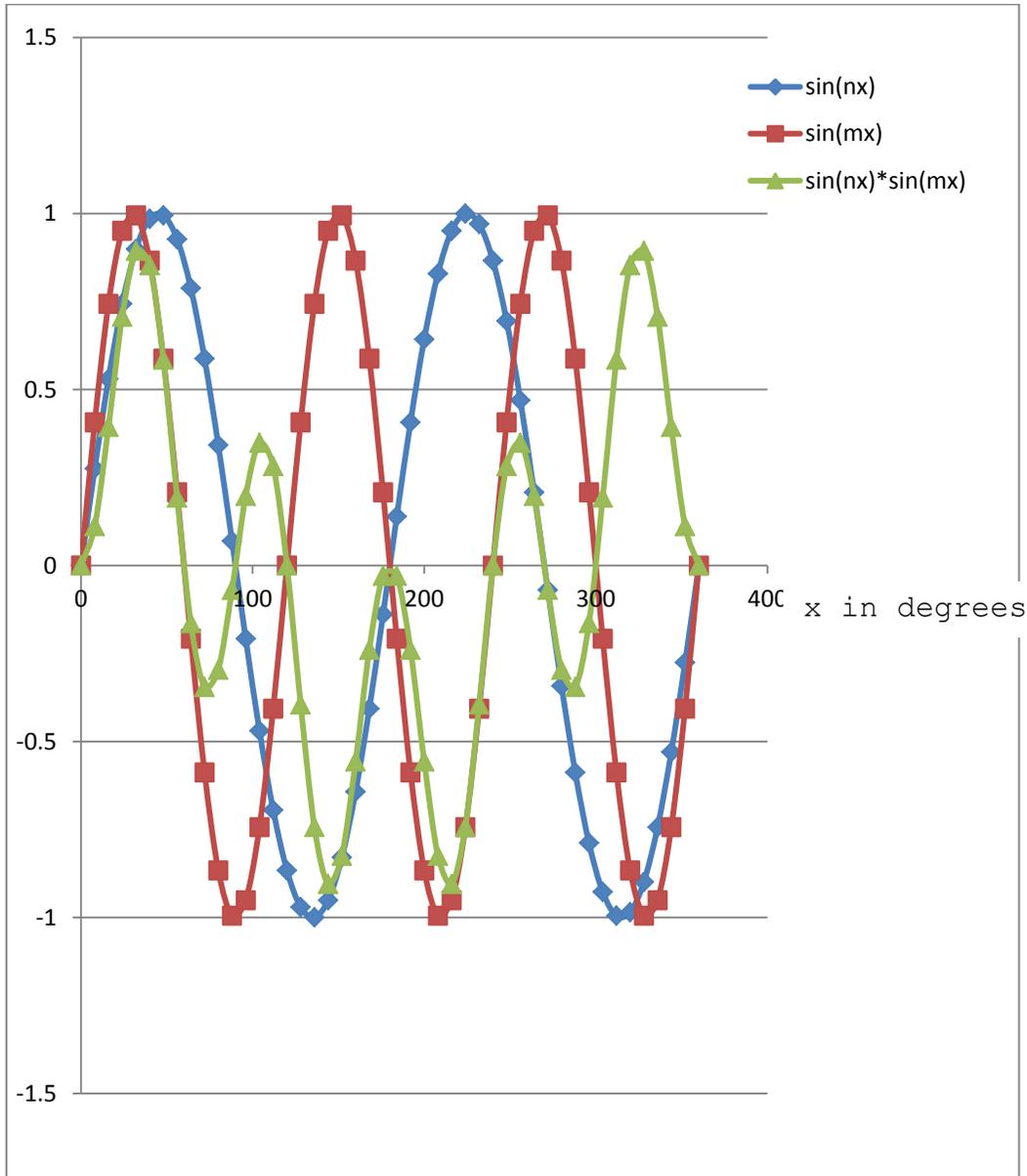
This can be generalized to:

$$C_n = (1/2\pi) \int_{\theta_0}^{\theta_0+2\pi} f(\theta) e^{-in\theta} d\theta \quad (\text{for all } n, \text{ both } + \text{ and } -).$$

We will use this form in the next sections. Note: the A_n 's, B_n 's, and $f(\theta)$ are real, while the C_n 's and $e^{i\theta}$ are complex.

It seems strange that real quantities can be expressed with complex numbers - but it works! What we find is that real, measurable quantities end up having imaginary components that equal zero.

Example of the product of two sine waves cancelling



The blue curve is $\sin(2x)$ and the red curve is $\sin(3x)$. The green curve shows the product of the two curves.

Note that from 0° to 180° the blue curve makes once complete oscillation, and the red curve makes 1.5 oscillations. From 180° to 360° the blue curve again makes one complete oscillation, and the red curve again makes 1.5 oscillations but in exactly the **opposite** direction. Therefore the values of $\sin(2x)\sin(3x)$ from 0° to 360° add up to zero. You can see from the green curve how there is as much green curve above zero as there is below zero.

To get the excel file that created the above graph download the excel spreadsheet from the course web page.