Appendix: Average Energy and Temperature

Consider a box with a single monatomic atom of mass, m, moving at some speed in the x direction, v_x. The length of the box in the x direction is L_x .

1. When the atom collides with the wall of the box, the collision causes the atom to change its momentum from mv_x to $-mv_x$ if we assume an elastic collision (no energy lost to the box or gained from the box).

2. According to Newton's 2nd Law, $\Sigma F_x = ma_x = dp_x/dt$. Therefore, by Newton's 3rd Law, the average force on the wall by the colliding atom will be equal and opposite to the average force by the wall on the atom which is equal to the change in momentum of the atom divided by the average time between collisions with the wall: $F_{x\text{-avg}} = \Delta p_x/\Delta t = 2mv_x/\Delta t$.

3. Since $v_x = \Delta x/\Delta t$, we get $\Delta t = \Delta x/v_x$. The average time between collisions is equal to the average distance from one wall to the opposite wall and back again, $2L_x$, divided by the speed, v_x, so $\Delta t = 2L_x/v_x$.

4. This gives $F_{x\text{-avg}} = \Delta p_x/\Delta t = (2mv_x) / (2L_x/v_x) = mv_x^2/L_x$.

5. If we divide both sides of the above equation by the perpendicular area, A_{yz}, we get $F_{x\text{-avg}} / A_{yz} = mv_x^2/(L_x A_{yz})$; the left side is just the pressure $F_{x\text{-avg}} / A_{yz} = P$; the denominator of the right side is the volume of the box ($L_x A_{yz}=V$). Moving the denominator of the right side to the left gives: $PV = mv_x^2$.

6. From the fact that (using < > for average), $\langle v^2\rangle = \langle v_x^2 + v_y^2 + v_z^2\rangle = \langle v_x^2\rangle+\langle v_y^2\rangle+\langle v_z^2\rangle$ and $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$, we get $\langle v_x^2\rangle = ⅓\langle v^2\rangle$, so we now have $PV = ⅓m\langle v^2\rangle = ⅔\langle ½mv^2\rangle = ⅔$ average kinetic energy.

7. We now **define the temperature** as being proportional to the average kinetic energy with a constant of proportionality = $(3/2)k_B$ to get **$(3/2)k_BT = \langle KE\rangle$** so that we now have $PV = k_BT$.

8. If we have N non-interacting particles in the box, the pressure is increased by a factor of N and so we have: $PV = Nk_BT$; we define the number of moles as $n = N/N_A$ where N_A is Avagadro's number = 6.02×10^{23} atoms/mole; this gives the ideal gas law, **$PV = nRT$** where $R = N_Ak_B$, the gas constant = 8.31 Joules/mole-Kelvin .

9. We recognize that there are three degrees of freedom for an atom in 3-D (one in x, one in y, one in z), we get the condition that **$½k_BT$ = average energy per degree of freedom**.

10. We find experimentally that the average energy in each available degree of freedom is the same as in any other available degree of freedom = $½k_BT$. This observation is called the **Equipartition of Energy**. Note that vibrational modes have a pair: kinetic and potential, and E&M waves (light) have a pair: E and B waves, so their average energy per mode is kT.