CHAPTER 14

Vector Calculus

1. Vector Fields

**DEFINITION.** A vector field in the plane is a function \( \mathbf{F}(x, y) \) from \( \mathbb{R}^2 \) into \( V_2 \). We write
\[
\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.
\]
A vector field in space is a function \( \mathbf{F}(x, y, z) \) from \( \mathbb{R}^3 \) into \( V_3 \), We write
\[
\mathbf{F}(x, y, z) = \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}.
\]

**EXAMPLE.** \( \mathbf{F}(x, y) = \langle -y, x \rangle = -y\mathbf{i} + x\mathbf{j} \)

Some values:
\[
\mathbf{F}(0, 1) = \langle -1, 0 \rangle = -\mathbf{i} \implies \|\mathbf{F}(0, 1)\| = 1.
\]
\[
\mathbf{F}(-1, -1) = \langle 1, -1 \rangle = \mathbf{i} - \mathbf{j} \implies \|\mathbf{F}(-1, -1)\| = \sqrt{2}.
\]
In general,
\[
\|\mathbf{F}(x, y)\| = \|\langle -y, x \rangle\| = \sqrt{x^2 + y^2}.
\]
The diagram on the left on the previous page is drawn to scale, with $\mathbf{F}(x, y)$ placed at $(x, y)$. The diagram on the right is scaled smaller with relative magnitudes to fit in the diagram.

This vector field could be called a “spin” field. Since

$$\langle x, y \rangle \cdot \langle -y, x \rangle = -xy + xy = 0,$$

each $\mathbf{F}(x, y)$ is tangent to the circle centered at the origin of radius $\sqrt{x^2 + y^2}$, pointing in a counter-clockwise direction with magnitude equalling the radius of the circle.
Sample vector fields:

1. $xi + yj$

Each vector has magnitude equal to the distance from its base to the origin.

2. $yi - xj$

Tangent to circles of radius $\sqrt{x^2 + y^2}$, clockwise magnitude = radians.

3. $\frac{xi + yj}{(x^2 + y^2)^{1/2}}$

Same as above, except each vector is a unit vector.

4. $y^2 i + x^2 j$

Point on x-axis, vector points up; point on y-axis, vector points right; elsewhere, vector points to some point in 1st quadrant.
More examples of scaled and unscaled vector fields:

1. \( \mathbf{F}(x, y) = x \mathbf{i} + x^2 \mathbf{j} \)

Scaled

Unscaled

2. \( \mathbf{F}(x, y) = x^2 \mathbf{i} + y^2 \mathbf{j} \)

Scaled

Unscaled

3. \( \mathbf{F}(x, y) = -y \mathbf{i} - x \mathbf{j} \)

Scaled

Unscaled

Motion above \( y \)-axis is to the right, below to the left.

Motion to right of \( y \)-axis is downward, to the left upward.

Right

Left
Some vector fields are velocity vector fields, i.e., $\mathbf{F}(x, y)$ gives the velocity of a particle at $(x, y)$. Suppose a particle starts to flow at $(x_0, y_0)$ at time $t_0$. Then the curve traced out by $\langle x(t), y(t) \rangle$, where $x(t)$ and $y(t)$ are solutions of the differential equations

$$x'(t) = f_1 (x(t), y(t)) \quad \text{and} \quad y'(t) = f_2 (x(t), y(t))$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, is a flow line.

![Vector field with a flow line through (1, 2)](image)

We use the chain rule to find a differential equation for $y$ as a function of $x$ for the velocity vector field:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{f_2 (x, y)}{f_1 (x, y)}.$$

In our example,

$$\frac{dy}{dx} = \frac{4}{3} \implies y = \frac{4}{3}x + C.$$

For the flow line through $(1, 2)$, $2 = \frac{4}{3} + C \implies C = \frac{2}{3}$. Thus $y = \frac{4}{3}x + \frac{2}{3}$ is the equation of the flow line.
Example. We consider the vector field $\mathbf{F}(x, y) = \langle 1, x \rangle$ and the flow line through $(-2, 2)$.

This is a velocity vector field for $\frac{dy}{dx} = \frac{x}{1} = x$. Then $y = \frac{1}{2}x^2 + C$. For the flow line through $(-2, 2)$, $2 = 2 + C \implies C = 0$. Thus the equation of the flow line is $y = \frac{1}{2}x^2$.

Definition. For any scalar function $f$ (from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}$), the vector field $\mathbf{F}(x, y) = \nabla f$ is called the gradient field for the function $f$. We call $f$ a potential function for $\mathbf{F}$. Whenever $\mathbf{F} = \nabla f$ for some scalar function $f$, we refer to $\mathbf{F}$ as a conservative vector field.
Gradient Fields and Level Curves

Compute the gradient fields for the following functions, and draw level curves \( f(x, y) = k \) for the indicated values of \( k \). Then sketch the gradient vector field at one or two points on each of these level curves.

1. \( f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}; k = 1, 2, 4 \)

\[ \nabla f(x, y) = \left( \frac{x}{2}, \frac{y}{3} \right) \]

\[ \frac{x^2}{4} + \frac{y^2}{9} = k \]

- \( k = 1 \):
  \[ \frac{x^2}{4} + \frac{y^2}{9} = 1 \]
- \( k = 2 \):
  \[ \frac{x^2}{8} + \frac{y^2}{18} = 1 \]
- \( k = 4 \):
  \[ \frac{x^2}{16} + \frac{y^2}{36} = 1 \]

2. \( f(x, y) = \frac{y}{x+y}; y \neq -x; k = \frac{1}{2}, \frac{3}{4}, 2 \)

\[ \nabla f(x, y) = \left( -\frac{y}{(x+y)^2}, \frac{x}{(x+y)^2} \right) \]

\[ \frac{y}{x+y} = k \]

- \( k = \frac{1}{2} \):
  \[ y = x \]
- \( k = \frac{3}{4} \):
  \[ y = 3x \]
- \( k = 2 \):
  \[ y = -x \]

In general, the further from the origin, the shorter the vector.
Problem (Page 996 #36). Determine whether

\[ \mathbf{F}(x, y) = \langle y \cos x, \sin x - y \rangle \]

is conservative, and if so, find its potential function.

(We proceed in a manner different from the text.) This is the same as determining whether

\[ M = f_x = y \cos x \quad N = f_y = \sin x - y \]

is exact and finding its solution as we did in Math 231.

\[ M_y = \cos x \quad N_x = \cos x \quad \rightarrow \quad M_y = N_x \quad \rightarrow \]

the DE is exact \( \quad \rightarrow \quad \mathbf{F} \) is conservative.

\[ f(x, y) = \int y \cos x \, dx \quad f(x, y) = \int (\sin x - y) \, dy \]

\[ = y \sin x + g(y) \quad = y \sin x - \frac{y^2}{2} + h(x) \]

\[ f_y(x, y) = \sin x + g'(y) \quad f_x(x, y) = y \cos x + h'(x) \]

\[ = \sin x - y \quad \rightarrow \quad = y \cos x \quad \rightarrow \]

\[ g'(y) = -y \quad \rightarrow \quad g(y) = -\frac{y^2}{2} + C \quad h'(x) = 0 \quad \rightarrow \quad h(x) = C \]

Thus

\[ f(x, y) = y \sin x - \frac{y^2}{2} + C \]

2. Line Integrals

Oriented curve — one from which we have chosen a direction — two possible directions.

**Definition.** The line integral of \( f(x, y, z) \) with respect to arc length along the oriented curve \( C \) in three-dimensional space, written \( \int_C f(x, y, z) \, ds \), is defined by

\[
\int_C f(x, y, z) \, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i,
\]

provided the limit exists and is the same for all choices of evaluation points.

**Note.** There is a similar definition for two dimensions.
**Theorem (Evaluation Theorem).** Suppose that \( f(x, y, z) \) is continuous in a region \( D \) containing the curve \( C \) and that \( C \) is described parametrically by \((x(t), y(t), z(t))\) for \( a \leq t \leq b \) where \( x(t) \), \( y(t) \), and \( z(t) \) all have continuous first derivatives. Then

\[
\int_{C} f(x, y, z) \, ds = \int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]

In the two-dimensional case,

\[
\int_{C} f(x, y) \, ds = \int_{a}^{b} f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.
\]

**Definition.** A space curve \( C \) is smooth if it can be described parametrically by \( x = x(t) \), \( y = y(t) \), and \( z = z(t) \) for \( a \leq t \leq b \), where \( x(t) \), \( y(t) \), and \( z(t) \) all have continuous first derivatives and \([x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2 \neq 0\) on \([a, b]\). (Similarly for plane curves.)

**Example.** Find \( \int_{C} 3y^2 \, ds \) where \( C \) is the quarter-circle \( x^2 + y^2 = 4 \) from \((0, 2)\) to \((-2, 0)\).

\[
x = 2 \cos t \implies x'(t) = -2 \sin t \quad \text{and} \quad y = 2 \sin t \implies y'(t) = 2 \cos t.
\]

\[
\int_{C} 3y^2 \, ds = \int_{\pi/2}^{\pi} [3(4 \sin^2 t)] \sqrt{4 \sin^2 t + 4 \cos^2 t} \, dt = 24 \int_{\pi/2}^{\pi} \sin^2 t \, dt = 12 \int_{\pi/2}^{\pi} (1 - \cos 2t) \, dt = 12 \left[ t - \frac{1}{2} \sin 2t \right]_{\pi/2}^{\pi} = 12 \left[ (\pi - 0) - \left( \frac{\pi}{2} - 0 \right) \right] = 6\pi.
\]
Problem (Page 1010 # 4). Find $\int_C xz\,ds$, where $C$ is the line segment from $(2, 1, 0)$ to $(2, 0, 2)$.

$x = 2$, and for $0 \leq t \leq 1$,

$$y = 1 - t, \quad z = 2t \implies x'(t) = 0, \quad y'(t) = -1, \quad z'(t) = 2.$$ 

$$ds = \sqrt{0^2 + (-1)^2 + 2^2} \, dt = \sqrt{5} \, dt$$

Thus

$$\int_C xz \, ds = \int_0^1 2(2t)\sqrt{5} \, dt = 4\sqrt{5} \int_0^1 t \, dt = 4\sqrt{5} \left[ \frac{t^2}{2} \right]_0^1 = 4\sqrt{5} \left( \frac{1}{2} - 0 \right) = 2\sqrt{5}.$$

Theorem. Suppose $f(x, y, z)$ is a continuous function in some region $D$ containing the oriented curve $C$. Then, if $C$ is piecewise smooth, with $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where $C_1, C_2, \ldots, C_n$ are all smooth and where the terminal point of $C_i$ is the same as the initial point of $C_{i+1}$, for $i = 1, 2, \ldots, n - 1$, we have

$$\int_{-C} f(x, y, z) \, ds = \int_C f(x, y, z) \, ds$$

and

$$\int_C f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds + \cdots + \int_{C_n} f(x, y, z) \, ds.$$

Note. There is a similar statement for two dimensions.
Example. Find $\int_C (x + y) \, ds$ over $C = C_1 \cup C_2$ where $C_1$ is the quarter circle from $(1, 0)$ to $(0, 1)$ and $C_2$ is the line segment from $(0, 1)$ to $(-1, 0)$.

\[ C_1: \, x(t) = \cos t, \, y(t) = \sin t, \, 0 \leq t \leq \frac{\pi}{2} \implies x'(t) = -\sin t, \, y'(t) = \cos t. \]

\[ C_2: \, x(t) = -t, \, y(t) = 1 - t, \, 0 \leq t \leq 1, \, x'(t) = -1, \, y'(t) = -1. \]

Thus
\[
\int_C (x + y) \, ds = \int_{C_1} (x + y) \, ds + \int_{C_2} (x + y) \, ds = \\
\int_0^{\pi/2} (\cos t + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt + \int_0^1 \left[ -t + (1 - t) \right] \sqrt{(-1)^2 + (-1)^2} \, dt = \\
\int_0^{\pi/2} (\cos t + \sin t) \, dt + \sqrt{2} \int_0^1 (1 - 2t) \, dt = \\
\left[ \sin t - \cos t \right]_0^{\pi/2} + \sqrt{2} \left[ t - t^2 \right]_0^1 = 1 + 1 + \sqrt{2}(0 - 0) = 2.
\]

Theorem. For any piecewise smooth curve $C$ (in two or three dimensions), $\int_C 1 \, ds$ gives the arc length of the curve $C$. 
Line integrals with respect to $x$

\[ \int_C f(x, y, z) \, dx = \lim_{\|p\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta x_i \]

\[ \int_{-C} f(x, y, z) \, dx = - \int_C f(x, y, z) \, dx \]

\[ \int_C f(x, y, z) \, dx = \int_{C_1} f(x, y, z) \, dx + \int_{C_2} f(x, y, z) \, dx + \cdots + \int_{C_n} f(x, y, z) \, dx \]

**Note.** We have similar results for $y$ and $z$ and two dimensions.

**Notation.** We write

\[ \int_C f(x, y, z) \, dx + \int_C g(x, y, z) \, dy + \int_C h(x, y, z) \, dz = \int_C f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz \]

**Example.** Find $\int_C 3y^2 \, dy$ where $C$ is the line segment from $(2, 0)$ to $(1, 3)$.

Parameterize the line by $x = 2 - t, y = 3t, 0 \leq t \leq 1$. Then $dy = 3 \, dt$ and

\[ \int_C 3y^2 \, dy = \int_0^1 3(3t)^2(3dt) = 81 \int_0^1 t^2 \, dt = 27t^3 \bigg|_0^1 = 27. \]
**Example.** Find $\int_C 3y^2\, dy$ where $C$ is the portion of $y = x^2$ from $(2, 4)$ to $(0, 0)$.

1) Parameterize by $x = -t, y = t^2$ from $-2 \leq t \leq 0$. Then $dy = 2t\, dt$ and

$$\int_C 3y^2\, dy = \int_{-2}^{0} 3t^4(2t)\, dt = 6 \int_{-2}^{0} t^5\, dt = t^6\bigg|_{-2}^{0} = 0 - (-2)^6 = -64$$

2) Could also parameterize by $x = 2 - t, y = (2 - t)^2$ from $0 \leq t \leq 2$. Then $dy = -2(2 - t)\, dt$ and

$$\int_C 3y^2\, dy = \int_{0}^{2} 3(2 - t)^4(-2)(2 - t)\, dt =$$

$$-6 \int_{0}^{2} (2 - t)^5\, dt = (2 - t)^6\bigg|_{0}^{2} = 0 - 64 = -64.$$  

**Line Integrals of Vector Fields**

Let $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ be a vector field along the curve $C$ defined by $x = x(t), y = y(t),$ and $z = z(t), a \leq t \leq b.$ Let

$$\mathbf{r} = \langle x, y, z \rangle \implies d\mathbf{r} = \langle dx, dy, dz \rangle.$$  

Define the line integral

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_C F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz =$$

$$\int_C F_1(x, y, z)\, dx + \int_C F_2(x, y, z)\, dy + \int_C F_3(x, y, z)\, dz$$  

**Work**

If $\mathbf{F}(x, y, z)$ is a force field, the work done by $\mathbf{F}$ in moving a particle along the curve $C$ can be written as

$$W = \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}.$$
**Example.** Find the work done by \( \mathbf{F}(x, y, z) = \langle z, 0, 3x^2 \rangle \) along \( C \), the quarter ellipse given by \( x = 2 \cos t, \ y = 3 \sin t, \ z = 1 \), from \( (2, 0, 1) \) to \( (0, 3, 1) \).

We have

\[
0 \leq t \leq \frac{\pi}{2}, \quad dx = -2 \sin t \, dt, \quad dy = 3 \cos t \, dt, \quad dz = 0.
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C z \, dx + 0 \, dy + F_3(x, y, z) \, dz = \int_0^{\pi/2} \left[ (1)(-2 \sin t) + 0(3 \cos t) + 3(4 \cos^2 t)(0) \right] \, dt = 2 \cos t \bigg|_{0}^{\pi/2} = 0 - 2 = -2.
\]

- Consider the vector field \( \mathbf{F}(x, y) \) and the curves \( C_1 \) and \( C_2 \) shown below. Explain why \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} > 0 \) and \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} < 0 \).

- Explain why \( \int_C \mathbf{F} \cdot d\mathbf{r} > 0 \) and \( \int_C \mathbf{G} \cdot d\mathbf{r} < 0 \) in the diagram below.

Recall \( \mathbf{F} \cdot d\mathbf{r} = \|\mathbf{F}\| \cdot \|d\mathbf{r}\| \cos \theta \), where \( \theta \) is the angle between \( \mathbf{F} \) and \( d\mathbf{r} \), \( 0 \leq \theta \leq \pi \). Thus the line integral of a vector field measures the extent to which \( C \) is going with the vector field (+) or against it (−).
Using the diagram below, arrange $\int_{C_i} \mathbf{F} \cdot d\mathbf{r}$, $i = 1 \ldots 4$, and the number 0 in order from left to right (smallest to largest).

\[
\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}
\]

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} < \int_{C_1} \mathbf{F} \cdot d\mathbf{r} < 0 < \int_{C_3} \mathbf{F} \cdot d\mathbf{r} < \int_{C_4} \mathbf{F} \cdot d\mathbf{r}
\]

$C_1$ and $C_2$ have the opposite direction of the vector field with the vectors on $C_2$ having the greater magnitude. $C_3$ and $C_4$ have the same direction of the vector field with the vectors having a similar magnitude, but $C_4$ is longer.

**Maple.** See lineintegral(14.2).mw or lineintegral(14.2).pdf.
3. Independence of Path and Conservative Vector Fields

Definition.

1. A region \( D \subseteq \mathbb{R}^n \) (for \( n \geq 2 \)) is called \text{connected} if every pair of points in \( D \) can be connected by a piecewise-smooth curve lying entirely in \( D \).

\[
\text{connected} \quad \text{not connected}
\]

2. A \text{path} is a piecewise-smooth curve \( C \) traced out by the endpoint of the vector-valued function \( \mathbf{r}(t) \) for \( a \leq t \leq b \).

3. The line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is \text{independent of path} in the domain \( D \) if the integral is the same for every path contained in \( D \) that has the same beginning and end points.

Theorem. Suppose the vector field \( \mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle \) is continuous on the open, connected region \( D \subseteq \mathbb{R}^2 \). Then the line integral

\[
\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}
\]

is independent of path if and only if \( \mathbf{F} \) is conservative on \( D \).

Note. A similar result is valid in any number of dimensions.
**THEOREM** (Fundamental Theorem for Line Integrals).

Suppose \( \mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle \) is continuous on the open, connected region \( D \subseteq \mathbb{R}^2 \) and \( C \) is any piecewise-smooth curve lying in \( D \), with initial point \((x_1, y_1)\) and terminal point \((x_2, y_2)\). Then, if \( \mathbf{F} \) is conservative on \( D \), with \( \mathbf{F}(x, y) = \nabla f(x, y) \),

\[
\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(x, y) \bigg|_{(x_2,y_2)}^{(x_1,y_1)} = f(x_2, y_2) - f(x_1, y_1).
\]

**EXAMPLE.** Compute \( \int_C \langle ye^{xy}, xe^{xy} \rangle \cdot d\mathbf{r} \) for the curve \( C \) shown below.

We need to find \( f(x, y) \) such that

\[
\nabla f = \langle f_x, f_y \rangle = \langle ye^{xy}, xe^{xy} \rangle.
\]

Assume \( f_x = ye^{xy} \). Then

\[
f(x, y) = \int ye^{xy} \, dx = e^{xy} + g(y) \implies f_y(x, y) = xe^{xy} + g'(y).
\]

Then \( \nabla f = \mathbf{F} \) if \( g(y) = C \) for some constant \( C \). Choose \( C = 0 \implies f(x, y) = e^{xy} \).

Then \( \mathbf{F} \) is conservative on \( \mathbb{R}^2 \), so

\[
\int_C \langle ye^{xy}, xe^{xy} \rangle \cdot d\mathbf{r} = e^{xy} \bigg|_{(3,1)}^{(-1,-1)} = e^3 - e.
\]
**Definition.** A curve $C$ is closed if its two endpoints are the same. For $C$ defined by $x = g(t), y = h(t), a \leq t \leq b$, this means $(g(a), h(a)) = (g(b), h(b))$.

**Theorem.** Suppose $F$ is continuous on the open, connected region $D \subseteq \mathbb{R}^2$. Then $F$ is conservative on $D$ if and only if $\int_C F(x, y) \cdot dr = 0$ for every piecewise-smooth closed curve $C$ lying in $D$.

**Definition.** A region $D$ is simply-connected if every closed curve in $D$ encloses only points in $D$.

**Theorem.** Suppose $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply-connected region $D$. Then $\int_C M(x, y) \, dx + N(x, y) \, dy$ is independent of path in $D$ if and only if $M_y(x, y) = N_x(x, y)$ for all $(x, y)$ in $D$.

**Example.** $F(x, y) = \langle M(x, y), N(x, y) \rangle = \langle x - y, x - 2 \rangle$. $M_y = -1$ and $N_x = 1$. Thus

$$\int_C F(x, y) \cdot dr = \int_C (x - y) \, dx + (x - 2) \, dy$$

is not independent of path.
**Theorem (Conservative Vector Fields).** Suppose \( \mathbf{F}(x, y) = (M(x, y), N(x, y)) \) and \( M(x, y) \) and \( N(x, y) \) have continuous first partial derivatives on an open, simply-connected region \( D \subseteq \mathbb{R}^2 \). Then the following are equivalent:

1. \( \mathbf{F}(x, y) \) is conservative in \( D \).
2. \( \mathbf{F}(x, y) \) is a gradient field in \( D \), i.e., \( \mathbf{F}(x, y) = \nabla f(x, y) \) for some potential function \( f \) for all \( (x, y) \) in \( D \).
3. \( \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} \) is independent of path in \( D \).
4. \( \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0 \) for every piecewise-smooth closed curve \( C \) lying in \( D \).
5. \( M_y(x, y) = N_x(x, y) \) for all \( (x, y) \) in \( D \).

**Example.** Are the following vector fields conservative?

\[
\begin{array}{c}
\text{F is constant } \implies M_y = N_x = 0 \implies \text{conservative.}
\end{array}
\]
For $C$ a counter-clockwise circle centered at the origin, $\int_C \mathbf{F} \cdot d\mathbf{r} > 0 \implies$ not conservative.

No rotation $\implies M_y = N_x \implies$ conservative.

**Theorem** (Three Dimensions). Suppose the vector field $\mathbf{F}(x, y, z)$ is continuous on the open, connected region $D \subseteq \mathbb{R}^3$. Then $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ is independent of path in $D$ if and only if $\mathbf{F}$ is conservative in $D$, i.e., $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$ for some scalar function $f$ (a potential function for $\mathbf{F}$) for all $(x, y, z)$ in $D$. Further, for any piecewise-smooth curve $C$ lying in $D$ with initial point $(x_1, y_1, z_1)$ and terminal point $(x_2, y_2, z_2)$,

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(x, y, z) \bigg|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$
Example. Compute $\int_C \langle yz^2, xz^2, 2xyz \rangle \cdot dr$ for the curve $C$ shown below.

We need to find $f(x, y, z)$ such that

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz^2, xz^2, 2xyz \rangle.$$ 

Assume

$$f_x = yz^2 \implies f(x, y, z) = \int yz^2 \, dx = xyz^2 + g(y, z).$$

Then

$$f_y = xz^2 + g_y(y, z) \implies g_y(y, z) = 0 \implies g(y, z) = h(z).$$

Thus

$$f(x, y, z) = xyz^2 + h(z) \implies f_z = 2xyz + h'(z) \implies h'(z) = 0 \implies h(z) = C.$$ 

Take $C = 0$. Then $f(x, y, z) = xyz^2$ and $\nabla f = \langle yz^2, xz^2, 2xyz \rangle = F$, so $F$ is conservative on $\mathbb{R}^3$, and thus

$$\int_C \langle yz^2, xz^2, 2xyz \rangle \cdot dr = xyz^2 \bigg|_{(0,1,2)}^{(0,2,2)} = 0 - 0 = 0.$$
4. Green’s Theorem

**Definition.**

(1) A curve $C$ is **simple** if it does not intersect itself, except at the endpoints.

(2) A simple closed curve $C$ has **positive orientation** if the region $R$ enclosed by $C$ stays to the left of $C$ as the curve is traversed; a simple closed curve $C$ has **negative orientation** if the region $R$ enclosed by $C$ stays to the right of $C$ as the curve is traversed.
**Notation.** \( \int_C F(x, y) \cdot dr \) denotes a line integral along a simple closed curve \( C \) oriented in the positive direction.

**Theorem (Green’s Theorem).** Let \( C \) be a piecewise-smooth simple closed curve in the plane with positive orientation and let \( R \) be the region enclosed by \( C \). Suppose \( M(x, y) \) and \( N(x, y) \) are continuous and have continuous first partial derivatives in some open region \( D \), with \( R \subset D \). Then

\[
\int_C M(x, y) \, dx + N(x, y) \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.
\]

**Example.** Find \( \int_C (x^4 + 2y) \, dx + (5x + \sin y) \, dy \).

With \( C \) made up of 4 separate continuous curves (lines), direct calculation and parametrizing is cumbersome, so use Green’s Theorem.

\[
\int_C (x^4 + 2y) \, dx + (5x + \sin y) \, dy = \iint_R (5 - 2) \, dA = 3 \iint_R dA = 3(\text{area of } R) = 3(\sqrt{2})^2 = 3 \cdot 2 = 6
\]

by geometry.
Example. Find \( \int_{C} (x^2y) \, dx + (x^3 + 2xy^2) \, dy \).

Again, for simplicity, use Green’s Theorem.

\[
\int_{C} (x^2y) \, dx + (x^3 + 2xy^2) \, dy = \iint_{R} \left[(3x^2 + 2y^2) - x^2\right] \, dA = \int\int_{R} (2x^2 + 2y^2) \, dA = 2 \int\int_{R} (x^2 + y^2) \, dA = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{r^2} r^2 \, dr \, d\theta = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4 - \frac{1}{4}\right) \, d\theta = \frac{15}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \, d\theta = \frac{15}{2} \theta \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{15}{2} \pi.
\]

Notation. We use \( \partial R \) to refer to the boundary of the region \( R \), oriented in the positive direction.

Note. We can replace \( C \) by \( \partial R \) in Green’s Theorem.
**Example.** Suppose \( C \) is a piecewise smooth, simple closed curve enclosing the region \( R \). Then

\[
(1) \quad \oint_C x \, dy = \iint_R (1 - 0) \, dA = \iint_R dA.
\]

\[
(2) \quad \oint_C (-y) \, dx = \iint_R [0 - (-1)] \, dA = \iint_R dA.
\]

Therefore,

\[
\text{Area of } R = \iint_R dA = \frac{1}{2} \oint_C x \, dy - y \, dx.
\]

**Extending Green’s Theorem**

Consider \( R \) as below:

Green’s Theorem doesn’t apply since \( R \) is not simply connected. But we make two horizontal slits in \( R \), dividing \( R \) into two simply-connected regions \( R_1 \) and \( R_2 \).

Apply Green’s Theorem to \( R_1 \) and \( R_2 \) separately:

\[
\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \iint_{R_1} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA + \iint_{R_2} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA =
\]

\[
\oint_{\partial R_1} M(x, y) \, dx + N(x, y) \, dy + \oint_{\partial R_2} M(x, y) \, dx + N(x, y) \, dy =
\]
\[
\int_{C_1} M(x, y) \, dx + N(x, y) \, dy + \int_{C_2} M(x, y) \, dx + N(x, y) \, dy = \\
\text{Since the line integrals over the slits cancel}
\int_{C} M(x, y) \, dx + N(x, y) \, dy
\]

where \( C = C_1 \cup C_2 \).

**Note.** This procedure can be extended to any finite number of holes.

**Example.** Suppose \( \mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle \). Show that

\[
\int_{C} \mathbf{F}(x, y) \cdot d\mathbf{r} = 2\pi
\]

for every simple closed curve \( C \) enclosing the origin.

**Solution.** Green’s Theorem doesn’t apply since \( \mathbf{F}(0, 0) \) is not defined. Let \( C \) be any simple closed curve containing the origin and let \( C_1 \) be the circle of radius \( a > 0 \) centered at the origin, positively oriented, where \( a \) is sufficiently small so that \( C \) and \( C_1 \) do not meet. Let \( R \) be the region between and including the curves.

![Diagram of a region with a simple closed curve and a circle enclosing it.](image)
Applying the extended Green’s Theorem,

\[ \oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} - \oint_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} = \oint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \]

\[ \iint_R \left[ \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} - \frac{(-1)(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} \right] dA = \iint_R 0 \, dA = 0 \implies \]

\[ \oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r}. \]

Parameterize \( C_1 \) by

\[ x = a \cos t, \quad y = a \sin t, \quad 0 \leq t \leq 2\pi. \]

Then \( x^2 + y^2 = a^2 \), and

\[ \oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} = \oint_{C_1} \left\langle \frac{-y}{a^2}, \frac{x}{a^2} \right\rangle \cdot d\mathbf{r} = \]

\[ \frac{1}{a^2} \oint_{C_1} (-y) \, dx + x \, dy = \frac{1}{a^2} \int_0^{2\pi} \left[ (a \sin t)(-a \sin t) + (a \cos t)(a \cos t) \right] \, dt = \]

\[ \int_0^{2\pi} \left( \sin^2 t + \cos^2 t \right) \, dt = \int_0^{2\pi} \, dt = t \bigg|_0^{2\pi} = 2\pi. \]

\( \square \)