We denote the directed line segment from the point \( P \) (initial point) to the point \( Q \) (terminal point) as \( \overrightarrow{PQ} \). The length of \( \overrightarrow{PQ} \) is its magnitude, denoted \( \| \overrightarrow{PQ} \| \).

**Definition.** A **vector** is the equivalence class of all directed line segments (or arrows) having the same magnitude and the same direction (an arrow and its parallel translates). Any arrow in the class may be used to name the vector.

**Example.**

\[
\begin{align*}
a &= b &= c
\end{align*}
\]

since \( a, b, c \) name the same vector.
**Definition.** A position vector is the arrow in each class with tail (initial point) at the origin. If the tip (terminal point) is at the point \( A(a_1, a_2) \), we write

\[
\mathbf{a} = \overrightarrow{OA} = \langle a_1, a_2 \rangle.
\]

In \( \langle a_1, a_2 \rangle \), \( a_1 \) and \( a_2 \) are called the **components** of \( \mathbf{a} \)— \( a_1 \) is the **first component**, \( a_2 \) the **second component**.

From the Pythagorean Theorem,

\[
\| \mathbf{a} \| = \| \langle a_1, a_2 \rangle \| = \sqrt{a_1^2 + a_2^2}.
\]

**Note.** The magnitude of \( \mathbf{a} \) is often called the **norm** or **2-norm** of \( \mathbf{a} \), and is sometimes denoted \( \| \mathbf{a} \|_2 \).

For \( \mathbf{a} = \overrightarrow{OA} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \overrightarrow{OB} = \langle b_1, b_2 \rangle \),

\[
\mathbf{a} = \mathbf{b} \iff A = B \iff a_1 = b_1 \text{ and } a_2 = b_2.
\]

\( \iff \) means “if and only if” (\( \implies \) is “only if” and \( \impliedby \) is “if”).
Basic Operations

Addition:

\[ \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle. \]

This is the tip-to-tail or parallelogram law. The sum \( \mathbf{a} + \mathbf{b} \) is called the resultant.

**Example.**

\[ \langle 4, 3 \rangle + \langle -1, 4 \rangle = \langle 3, 7 \rangle \]
Subtraction:

\[ \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle. \]

For direction of difference vector, make sure \( b + (a - b) = a \).

Zero vector:

\[ 0 = \langle 0, 0 \rangle \]

\( \|0\| = 0 \) and the zero vector has all directions.

Scalar multiplication:

\[ ca = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle \]

Thus \( ca \) is a vector and

\[ \|ca\| = \|\langle ca_1, ca_2 \rangle\| = \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2a_1^2 + c^2a_2^2} = \sqrt{c^2(a_1^2 + a_2^2)} = c \cdot \|a\|. \]
1. VECTORS IN THE PLANE

Vectors are parallel if one is a scalar multiple of the other. $ca \parallel a$, in the same direction for $c \geq 0$, the opposite direction for $c \leq 0$.

Additive inverse:

$$-a = -(a_1, a_2) = (-1)(a_1, a_2) = (-a_1, -a_2)$$

$$\| -a \| = \| (-1)(a_1, a_2) \| = | -1 | \cdot \| (a_1, a_2) \| = \| a \|$$

**Definition.** $V_2 = \{ (x, y) | x, y \in \mathbb{R} \}$, the set of all two-dimensional position vectors.

**Definition.** A vector space is a set of vectors with the same dimension and a set of scalars (real or complex numbers) with addition and scalar multiplication such that for any vectors $u, v,$ and $w$, and any scalars $\alpha$ and $\beta$,

1. $v + w = w + v$
2. $(u + v) + w = u + (v + w)$
3. $\alpha(\beta v) = (\alpha \beta)v$
4. $(\alpha + \beta)v = \alpha v + \beta v$
5. $\alpha(v + w) = \alpha v + \alpha w$
6. $1v = v$
7. $0v = 0$
8. $v + 0 = v$
9. For each $v$ there is $-v$ such that $v + (-v) = 0$
**Theorem (1.1).** $V_2$ is a vector space.

**Proof.** (of (2))

\[
(u + v) + w = (u_1 + v_1, u_2 + v_2) + (w_1, w_2) = \\
(u_1 + v_1 + w_1, u_2 + v_2 + w_2) = \\
(u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) = \\
(u_1 + v_1, u_2) + (v_1 + w_1, v_2 + w_2) = \\
(u_1, u_2) + (v_1, v_2) + (w_1, w_2) = u + (v + w)
\]

**Note.** For any two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$, $\overrightarrow{AB}$ corresponds to the position vector $\langle x_2 - x_1, y_2 - y_1 \rangle$.

Any vector can be written in terms of the standard basis vectors

\[
i = \langle 1, 0 \rangle \quad \text{and} \quad j = \langle 0, 1 \rangle.
\]

For any $a \in V_2$,

\[
a = \langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle = a_1 i + a_2 j.
\]

We call $a_1$ and $a_2$ the horizontal and vertical components of $a$

**Note.** $\|i\| = \|j\| = 1$.

A **unit vector** is a vector $a$ where $\|a\| = 1$.

**Theorem (1.2).** For any nonzero vector $a$, a unit vector having the same direction as $a$ is

\[
u = \frac{1}{\|a\|} a.
\]

**Proof.**

$a \neq 0 \implies \|a\| > 0 \implies u$ is a positive scalar multiple of $a \implies u$ has the same direction as $a$.

\[
\|u\| = \left\| \frac{1}{\|a\|} a \right\| = \left\| \frac{1}{\|a\|} \right\| \cdot \|a\| = \frac{1}{\|a\|} \cdot \|a\| = 1.
\]
**Example.** Find a unit vector in the same direction as \( \mathbf{a} = \langle 6, 8 \rangle \).

\[
\|\mathbf{a}\| = \|\langle 6, 8 \rangle\| = \sqrt{36 + 64} = \sqrt{100} = 10.
\]

Therefore

\[
\mathbf{u} = \frac{1}{10} \langle 6, 8 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.
\]

Resolving a vector \( \mathbf{a} \) into horizontal and vertical components:

\[
\mathbf{a} = \|\mathbf{a}\| \cos \theta \mathbf{i} + \|\mathbf{a}\| \sin \theta \mathbf{j}
\]

\[
= \|\mathbf{a}\| (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})
\]

\[
= \|\mathbf{a}\| \langle \cos \theta, \sin \theta \rangle
\]

**Note.** \( \|\langle \cos \theta, \sin \theta \rangle\| = 1. \)
Example. A plane is flying at 500 km/hr. The wind is blowing at 60 km/hr toward the SE. In what direction should the plane head to go due east? What is the plane’s speed relative to the ground?

Solution

Let \( \mathbf{p} \) = the motion vector for the plane and \( \mathbf{w} \) = the wind vector.

\[
\mathbf{w} = \|\mathbf{w}\| \cos(-45^\circ)\mathbf{i} + \|\mathbf{w}\| \sin(-45^\circ)\mathbf{j} = 60\frac{\sqrt{2}}{2}\mathbf{i} - 60\frac{\sqrt{2}}{2}\mathbf{j} = 30\sqrt{2}\mathbf{i} - 30\sqrt{2}\mathbf{j} = 42.43\mathbf{i} - 42.43\mathbf{j}.
\]

Since the \( \mathbf{j} \)-component of \( \mathbf{p} + \mathbf{w} \) is 0,

\[
\mathbf{p} = x\mathbf{i} + 30\sqrt{2}\mathbf{j}
\]

where \( x \) is the \( \mathbf{i} \)-component of \( \mathbf{p} \). Since \( \|\mathbf{p}\| = 500 \),

\[
500 = \sqrt{x^2 + (30\sqrt{2})^2} \implies 500^2 = x^2 + 1800 \implies x^2 = 250,000 - 1800 = 248,200 \implies x \approx 498.20.
\]

Thus

\[
\mathbf{p} = 498.20\mathbf{i} + 42.43\mathbf{j} \implies \tan \phi = \frac{42.43}{498.20} = .085 \implies \phi = \arctan .085 = 4.86^\circ,
\]

and since \( \mathbf{p} + \mathbf{w} = 540.63\mathbf{i} + 0\mathbf{j}, \|\mathbf{p} + \mathbf{w}\| = 540.63 \text{ km/hr}. \]
MAPLE. See \texttt{vectorsinplane(10.1).mw} or \texttt{vectorsinplane(10.1).pdf}

2. Vectors in Space

We use a \textbf{right-handed} coordinate system.

The coordinate system has 8 \textbf{octants}. The \textbf{first octant} has $x > 0$, $y > 0$, $z > 0$.

Distance Formula

$$d\{(x_1, y_1, z_1), (x_2, y_2, z_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$  

Vectors in $\mathbb{R}^3$

These are similar to vectors in $\mathbb{R}^2$, with one more dimension.

$$V_3 = \{ \langle x, y, z \rangle | x, y, z \in \mathbb{R} \}$$

is a vector space with

$$\mathbf{a} + \mathbf{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle,$$

$$c\mathbf{a} = c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle,$$

\text{and}

$$\mathbf{a} - \mathbf{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle,$$
Also,
\[ \|a\| = \|\langle a_1, a_2, a_3 \rangle\| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \]
so
\[ \|ca\| = |c|\|a\|, \quad 0 = \langle 0, 0, 0 \rangle, \text{ and} \]
\[ -a = \langle -a_1, -a_2, -a_3 \rangle. \]

**NOTE.** The vector with initial point \( P(a_1, a_2, a_3) \) and terminal point \( Q(b_1, b_2, b_3) \) corresponds to the position vector
\[ \overrightarrow{PQ} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle. \]

The standard basis consists of
\[ \mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle \text{ with} \]
\[ \|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1. \]
For any vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \),
\[ \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}. \]
Also, for any vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \neq 0 \), a unit vector \( \mathbf{u} \) in the same direction as \( \mathbf{a} \) is
\[ \mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a}. \]

The sphere of radius \( r \) centered at \( (a, b, c) \) consists of all points \( (x, y, z) \) such that
\[ d\{(x, y, z), (a, b, c)\} = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} \]
or
\[ (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2, \]
the standard form of the equation of a sphere.
**Problem (Page 711 #28).** Identify the geometric shape given by
\[ x^2 + x + y^2 - y + z^2 = \frac{7}{2}. \]

**Solution**

Rewrite as
\[ (x^2 + x) + (y^2 - y) + z^2 = \frac{7}{2}. \]

Complete squares:
\[
\left( x^2 + x + \frac{1}{4} \right) + \left( y^2 - y + \frac{1}{4} \right) + z^2 = \frac{7}{2} + \frac{1}{4} + \frac{1}{4} \implies \\
\left( x + \frac{1}{2} \right)^2 + \left( y - \frac{1}{2} \right)^2 + z^2 = 4
\]

Thus we have the sphere of radius 2 centered at \((-\frac{1}{2}, \frac{1}{2}, 0)\).

**Problem (Page 711 #20).** Find a vector of magnitude 3 in the same direction as \( \mathbf{v} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} \).

**Solution**

\[ \| \mathbf{v} \| = \sqrt{3^2 + 3^2 + (-1)^2} = \sqrt{19} \]

Thus
\[ \mathbf{u} = \frac{1}{\sqrt{19}} \mathbf{v} \]

is a unit vector in the same direction of \( \mathbf{v} \), so
\[ 3\mathbf{u} = \frac{3}{\sqrt{19}}\mathbf{v} = \frac{9}{\sqrt{19}}\mathbf{i} + \frac{9}{\sqrt{19}}\mathbf{j} - \frac{3}{\sqrt{19}}\mathbf{k} \]

has magnitude 3 in the same direction as \( \mathbf{v} \).

**Problem (Page 711 #40).** An equation for the \( x \)-axis is
\[ y^2 + z^2 = 0 \quad \text{or} \quad y = z = 0. \]
**Problem** (Page 712 #62). Three ropes are attached to a 300-pound crate. Rope $A$ exerts a force of $\langle 10, -130, 200 \rangle$ pounds on the crate, and rope $B$ exerts a force of $\langle -20, 180, 160 \rangle$ pounds on the crate. What force must rope $C$ exert on the crate to move it up and to the right with a constant force of $\langle 0, 30, 20 \rangle$ pounds.

**Solution**

Force of rope $A$ is $\mathbf{a} = \langle 10, -130, 200 \rangle$, rope $B$ is $\mathbf{b} = \langle -20, 180, 160 \rangle$, rope $C$ is $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, and the weight is $\mathbf{w} = \langle 0, 0, -300 \rangle$.

We want the net force

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{w} = \langle 0, 30, 20 \rangle \implies$$

$$\langle -10 + c_1, 50 + c_2, 60 + c_3 \rangle = \langle 0, 30, 20 \rangle \implies$$

$$c_1 = 10, \quad c_2 = -20, \quad c_3 = -40 \implies$$

$$\mathbf{c} = \langle 10, -20, -40 \rangle \implies \|\mathbf{c}\| \approx 45.8.$$  

Thus rope $C$ exerts a force of about 45.8 pounds in the direction $\langle 1, -2, -4 \rangle$.

**Maple.** See `vectorsinspace(10.2).mw` or `vectorsinspace(10.2).pdf`

### 3. The Dot Product

This is also often referred to as the **scalar product**.

In $V_2$:

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

In $V_3$:

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Example.** If $\mathbf{a} = \langle -3, 2, 5 \rangle$ and $\mathbf{b} = \langle 3, 5, -6 \rangle$,

$$\mathbf{a} \cdot \mathbf{b} = \langle -3, 2, 5 \rangle \cdot \langle 3, 5, -6 \rangle = (-3)(3) + 2(5) + 5(-6) = -9 + 10 - 30 = -29$$
Note.

(1) The dot product of two vectors is a number.

(2) A vector \( \langle a, b \rangle \) in \( V_2 \) can be thought of a vector \( \langle a, b, 0 \rangle \) in \( V_3 \), so results we prove for \( V_3 \) also hold for \( V_2 \).

**Theorem (3.1).** For vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and any scalar \( d \),

(1) \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)

(2) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)

(3) \( (d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b}) \)

(4) \( \mathbf{0} \cdot \mathbf{a} = 0 \)

(5) \( \mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \)

**Proof.**

(2) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \)

\[ = \langle a_1, a_2, a_3 \rangle \cdot (b_1 + c_1, b_2 + c_2, b_3 + c_3) \]

\[ = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \]

\[ = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \]

\[ = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 \]

\[ = \langle a_1, a_2, a_3 \rangle \cdot (b_1, b_2, b_3) + \langle a_1, a_2, a_3 \rangle \cdot (c_1, c_2, c_3) \]

\[ = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \]

(5) \( \mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = \|\langle a_1, a_2, a_3 \rangle\|^2 = \|\mathbf{a}\|^2. \)

Note. \( \mathbf{a} \cdot \mathbf{b} = 0 \) does not imply \( \mathbf{a} = 0 \) or \( \mathbf{b} = 0 \), so this differs from our usual multiplication.
For $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ in $V_3$, the angle $\theta$ ($0 \leq \theta \leq \pi$) between $\mathbf{a}$ and $\mathbf{b}$ is the smaller of the two angles in the plane formed by giving both vectors the same initial point.

If $\mathbf{a}$ and $\mathbf{b}$ have the same direction, $\theta = 0$.
If $\mathbf{a}$ and $\mathbf{b}$ have the opposite direction, $\theta = \pi$.
$\mathbf{a}$ and $\mathbf{b}$ are orthogonal (or perpendicular) if $\theta = \frac{\pi}{2}$.
We say $\mathbf{0}$ is orthogonal to every vector.

Law of Cosines

$$||\mathbf{a} - \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||||\mathbf{b}|| \cos \theta$$

**Theorem (3.2).** Let $\theta$ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta.$$ 

**Corollary.** For nonzero vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}.$$ 

**Example.** Find the angle between $\mathbf{a} = (2, 3, 6)$ and $\mathbf{b} = (-2, 1, 4)$.

$$\cos \theta = \frac{2(-2) + 3(1) + 6(4)}{\sqrt{4 + 9 + 36} \sqrt{4 + 1 + 16}} = \frac{23}{7\sqrt{21}} \Rightarrow$$

$$\theta = \arccos \left( \frac{23}{7\sqrt{21}} \right) \approx .7713^R \approx 44.19^\circ$$
3. THE DOT PRODUCT

**Corollary.** \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal \( (\mathbf{a} \perp \mathbf{b}) \iff \mathbf{a} \cdot \mathbf{b} = 0 \).

**Theorem (3.3 — Cauchy-Schwarz Inequality).** For any vectors \( \mathbf{a} \) and \( \mathbf{b} \),

\[
|\mathbf{a} \cdot \mathbf{b}| \leq ||\mathbf{a}|| ||\mathbf{b}| |
\]

or

\[
|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}.
\]

**Proof.**

\[
|\mathbf{a} \cdot \mathbf{b}| = ||\mathbf{a}|| ||\mathbf{b}| | \cos \theta = ||\mathbf{a}|| ||\mathbf{b}| | \cos \theta \leq ||\mathbf{a}|| ||\mathbf{b}| |.
\]

**Theorem (3.4 — Triangle Inequality).** For any vectors \( \mathbf{a} \) and \( \mathbf{b} \),

\[
||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||.
\]

**Proof.**

\[
||\mathbf{a} + \mathbf{b}||^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}
\]

\[
= ||\mathbf{a}||^2 + 2 \mathbf{a} \cdot \mathbf{b} + ||\mathbf{b}||^2
\]

\[
\leq ||\mathbf{a}||^2 + 2 ||\mathbf{a}|| ||\mathbf{b}|| + ||\mathbf{b}||^2
\]

\[
= (||\mathbf{a}|| + ||\mathbf{b}||)^2
\]

Taking square roots then gives the result.

---

**Diagram:**

- \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal.
- \( \mathbf{a} + \mathbf{b} \) is the resultant vector.
- The magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \) are represented by the lengths of the arrows.
- The angle between \( \mathbf{a} \) and \( \mathbf{b} \) is \( \theta \).
- The Pythagorean theorem is applied to the triangle formed by \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{a} + \mathbf{b} \).
Component of \( \mathbf{a} \) along \( \mathbf{b} \)

\[
\text{comp}_b \mathbf{a} \quad \left( 0 < \theta < \frac{\pi}{2} \right)
\]

\[
\text{-comp}_b \mathbf{a} \quad \left( \frac{\pi}{2} < \theta < \pi \right)
\]

**NOTE.** \( \cos(\pi - \theta) = -\cos \theta \)

\[
\text{comp}_b \mathbf{a} = \|\mathbf{a}\| \cos \theta = \frac{\|\mathbf{a}\| \|\mathbf{b}\|}{\|\mathbf{b}\|} \cos \theta = \frac{1}{\|\mathbf{b}\|} \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta = \frac{1}{\|\mathbf{b}\|} \mathbf{a} \cdot \mathbf{b} \implies \]

\[
\text{comp}_b \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}
\]

is a number, a directed length, that depends on the length of \( \mathbf{a} \) since \( \frac{\mathbf{b}}{\|\mathbf{b}\|} \) is a unit vector.
Projection of $\mathbf{a}$ onto $\mathbf{b}$

This is a force vector parallel to $\mathbf{b}$ having the same component along $\mathbf{b}$ as $\mathbf{a}$. 

$$\text{proj}_b \mathbf{a} \quad \left( 0 < \theta < \frac{\pi}{2} \right)$$ 

$$\text{proj}_b \mathbf{a} = \left( \text{comp}_b \mathbf{a} \right) \frac{\mathbf{b}}{||\mathbf{b}||} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \right) \frac{\mathbf{b}}{||\mathbf{b}||}$$ 

is a vector.
**PROBLEM** (Page 721 #63b). Find the component of the weight vector along the road vector for a 2500-pound car on a $15^\circ$ bank.

Let $\mathbf{b}$ be the direction of the banked road with $||\mathbf{b}|| = 1$.

\[
\mathbf{b} = \langle \cos 15^\circ, \sin 15^\circ \rangle.
\]

The weight

\[
\mathbf{w} = \langle 0, -2500 \rangle.
\]

The component of $\mathbf{w}$ in the direction of $\mathbf{b}$ is

\[
\text{comp}_b \mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{b}}{||\mathbf{b}||} = \langle 0, -2500 \rangle \cdot \langle \cos 15^\circ, \sin 15^\circ \rangle = -2500 \sin 15^\circ = -647.
\]

Thus the force is to the inside of the track, allowing the car to turn.

**Problem** (Page 720 #56). The **orthogonal projection** of a vector $\mathbf{a}$ along a vector $\mathbf{b}$ is defined as

\[
\text{orth}_b \mathbf{a} = \mathbf{a} - \text{proj}_b \mathbf{a}.
\]

Sketch a picture showing $\mathbf{a}$, $\mathbf{b}$, $\text{proj}_b \mathbf{a}$, and $\text{orth}_b \mathbf{a}$, and explain what is orthogonal about $\text{orth}_b \mathbf{a}$.

**Maple.** See dotprod(10.3).mw or dotprod(10.3).pdf
4. The Cross Product

This is a product that exists only in 3 dimensions. We first need an introduction to determinants for computational purposes.

**Definition.** For a $2 \times 2$ ("2 by 2") matrix $M = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, the determinant of $M$ is

$$|M| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

**Definition.** The determinant of a $3 \times 3$ matrix $M$ is

$$|M| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

**Example.**

$$\begin{vmatrix} 2 & -3 & 4 \\ 5 & 1 & -1 \\ 6 & -2 & 3 \end{vmatrix} = 6 - 4 + 18 + 45 - 40 - 24 = 1$$

Area of the Parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$

$$\text{Area} = (\text{base}) \cdot (\text{height}) = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$
**Definition.** The cross product \( \mathbf{a} \times \mathbf{b} \) is the vector orthogonal to \( \mathbf{a} \) and \( \mathbf{b} \) by the right-hand rule, whose magnitude is the area of the parallelogram determined by the vectors \( \mathbf{a} \) and \( \mathbf{b} \), i.e.,

\[
\| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \theta.
\]

**Theorem (4.4).** For nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( V_3 \), if \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \) \( (0 < \theta < \pi) \), then

\[
\| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \theta.
\]

**Theorem (4.2).** For any vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( V_3 \),

\( \mathbf{a} \times \mathbf{b} \perp \mathbf{a} \) and \( \mathbf{a} \times \mathbf{b} \perp \mathbf{b} \).

**Definition (Algebraic Definition of Cross Product).**

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2 b_3 \mathbf{i} - a_1 b_3 \mathbf{j} + a_1 b_2 \mathbf{k}.
\]
**Example.**

\[
\langle 5, 1, -2 \rangle \times \langle -1, 3, 4 \rangle = \begin{vmatrix} i & j & k \\ 5 & 1 & -2 \\ -1 & 3 & 4 \end{vmatrix} = (4 + 6)i + (2 - 20)j + (15 + 1)k = \langle 10, -18, 16 \rangle
\]

\[
\|\langle 10, -18, 16 \rangle\| = \sqrt{100 + 324 + 256} = \sqrt{680} = 2\sqrt{170} \approx 26.08.
\]

**Theorem (4.1).** For any vector \( \mathbf{a} \) in \( V_3 \),

\[ \mathbf{a} \times \mathbf{a} = \mathbf{0} \quad \text{and} \quad \mathbf{a} \times \mathbf{0} = \mathbf{0}. \]

**Example.**

\[
\mathbf{i} \times \mathbf{j} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k},
\]

but

\[
\mathbf{j} \times \mathbf{i} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k},
\]

so the cross product is not commutative.

- \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \)
- \( \mathbf{j} \times \mathbf{i} = -\mathbf{k} \)
- \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \)
- \( \mathbf{k} \times \mathbf{j} = -\mathbf{i} \)
- \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \)
- \( \mathbf{i} \times \mathbf{k} = -\mathbf{j} \)
THEOREM (4.3). For any \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in V_3 \), and any scalar \( d \),

\begin{enumerate}
\item \( \mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) \) (anti-commutativity)
\item \( (d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b}) \)
\item \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \)
\item \( (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \)
\item \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \) (scalar triple product)
\item \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \) (vector triple product)
\end{enumerate}

COROLLARY (to Theorem 4.4). Two nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( V_3 \) are parallel if and only if \( \mathbf{a} \times \mathbf{b} = \mathbf{0} \).

PROOF.

\( \mathbf{a} \parallel \mathbf{b} \iff \theta(\angle \text{ between } \mathbf{a} \text{ and } \mathbf{b}) = 0 \) or \( \theta = \pi \iff \sin \theta = 0 \iff \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin \theta = 0 \iff \mathbf{a} \times \mathbf{b} = \mathbf{0} \).

Volume of a Parallelepiped having noncoplanar vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) as adjacent edges.

\[
\text{Volume} = (\text{area of base}) \cdot \text{altitude} = \|\mathbf{a} \times \mathbf{b}\| \cdot |\text{comp}_{\mathbf{a} \times \mathbf{b}} \mathbf{c}| = \|\mathbf{a} \times \mathbf{b}\| \cdot \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{\|\mathbf{a} \times \mathbf{b}\|} = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| 
\]

the absolute value of a scalar triple product.
Corollary (to Theorem 4.3). The scalar triple product of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \]

Note.
\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
= \langle a_1, a_2, a_3 \rangle \cdot \left( b_2 \begin{vmatrix} b_3 & i \\ c_3 & c_1 \end{vmatrix} - b_1 \begin{vmatrix} b_3 & j \\ c_3 & c_2 \end{vmatrix} + b_1 \begin{vmatrix} b_2 & k \\ c_2 & c_1 \end{vmatrix} \right) \\
= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]

Example. Find the volume of the parallelepiped with 3 adjacent edges formed by $\langle -2, -1, 2 \rangle$, $\langle 1, -1, 2 \rangle$, and $\langle 2, 1, 3 \rangle$.
\[
V = \begin{vmatrix} -2 & -1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = |6 + 4 - 4 + 3 + 2 + 4| = 15
\]
Example. Find the distance from a point $Q$ to the line through $P$ and $R$.

\[
\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\| \sin \theta = \|\overrightarrow{PR}\| \cdot d \implies d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}
\]

Problem (Page 732 #20). Find the distance from the point $Q = (1, 3, 1)$ to the line through $P = (1, 3, -2)$ and $R = (1, 0, -2)$.

\[
\overrightarrow{PQ} = \langle 0, 0, 3 \rangle \quad \text{and} \quad \overrightarrow{PR} = \langle 0, -3, 0 \rangle \implies d = \frac{\|\langle 0, 0, 3 \rangle \times \langle 0, -3, 0 \rangle\|}{\|\langle 0, -3, 0 \rangle\|} = \frac{1}{3} \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3 \\ 0 & -3 & 0 \end{array} \right| = \frac{1}{3} \|9\mathbf{i}\| = \frac{1}{3}(9) = 3.
\]
**Example (Torque applied by a wrench).**

\[
\begin{align*}
\mathbf{F} &= \text{force applied at end of handle.} \\
\tau &= \text{torque applied to the bolt.} \\
\mathbf{r} &= \text{position vector for handle.} \\
\end{align*}
\]

Then

\[
\tau = \mathbf{r} \times \mathbf{F} \implies \|\tau\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| \sin \theta
\]

**Problem (Page 732 #28).** If you apply a force of magnitude 40 pounds at the end of an eighteen-inch-long wrench at an angle of \(\frac{\pi}{3}\) to the wrench, find the magnitude of the torque applied to the wrench.

\[
\begin{align*}
\|\mathbf{F}\| &= 40, \quad \|\mathbf{r}\| = 18, \quad \theta = \frac{\pi}{3} \implies \\
\|\tau\| &= 18 \cdot 40 \sin \frac{\pi}{3} = 720 \frac{\sqrt{3}}{2} = 360\sqrt{3} \approx 623.54.
\end{align*}
\]

**Maple.** See [crossprod(10.4).mw](#) or [crossprod(10.4).pdf](#).
5. Lines and Planes in Space

Lines in 3-space:

(1) Vector equation specified by two points or a point and a direction.

\[ P = \overrightarrow{OP}_1 + t\mathbf{a} = \langle x_1, y_1, z_1 \rangle + t\langle a_1, a_2, a_3 \rangle = \langle x_1 + ta_1, y_1 + ta_2, z_1 + ta_3 \rangle \]

or

\[ \overrightarrow{P_1P} \parallel \mathbf{a} \implies \overrightarrow{P_1P} = t\mathbf{a} \implies \langle x - x_1, y - y_1, z - z_1 \rangle = t\langle a_1, a_2, a_3 \rangle \]

**Note.** If given two points, use one point with the direction vector determined by the two points.

(2) Parametric equations

\[ x = x_1 + ta_1, \quad y = y_1 + ta_2, \quad z = z_1 + ta_3 \]

or

\[ x - x_1 = ta_1, \quad y - y_1 = ta_2, \quad z - z_1 = ta_3 \]

(3) Symmetric equations (solve for \( t \) in the three equations above, and equate the results).

\[ \frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3} \]
Problem (Page 740 #8). Find the vector, parametric and symmetric equations of the line through \((-3, 1, 0)\) and perpendicular to both \(\langle 0, -3, 1 \rangle\) and \(\langle 4, 2, -1 \rangle\).

\[
\langle 0, -3, 1 \rangle \times \langle 4, 2, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 1 \\ 4 & 2 & -1 \end{vmatrix} = \langle 1, 4, 12 \rangle
\]
is perpendicular to both \(\langle 0, -3, 1 \rangle\) and \(\langle 4, 2, -1 \rangle\). Then the vector equation is

\[
\langle x + 3, y - 1, z \rangle = t\langle 1, 4, 12 \rangle,
\]
the parametric equations are

\[
x + 3 = t, \quad y - 1 = 4t, \quad z = 12t,
\]
and the symmetric equations are

\[
\frac{x + 3}{1} = \frac{y - 1}{4} = \frac{z}{12}.
\]

For \(t = 3\), \(x = 0\), \(y = 13\), and \(z = 36\), so \((0, 13, 36)\) is on this line and each quotient of the symmetric equations is 3.

Definition. Let \(l_1\) and \(l_2\) be two lines in \(\mathbb{R}^3\), with parallel (or direction) vectors \(\mathbf{a}\) and \(\mathbf{b}\), respectively, and let \(\theta\) be the angle between \(\mathbf{a}\) and \(\mathbf{b}\).

1. \(l_1\) and \(l_2\) are parallel whenever \(\mathbf{a}\) and \(\mathbf{b}\) are parallel.
2. If \(l_1\) and \(l_2\) intersect, then
   
   a. the angle between \(l_1\) and \(l_2\) is \(\theta\).
   b. \(l_1\) and \(l_2\) are orthogonal whenever \(\mathbf{a}\) and \(\mathbf{b}\) are orthogonal.
3. Nonparallel, nonintersecting lines are called skew lines.
**Problem (Page 740 #14).** Determine whether the following lines are parallel, skew, or intersect:

\[
\begin{align*}
  x &= 1 - 2t \\
  y &= 2t \\
  z &= 5 - t
\end{align*}
\]

\[
\begin{align*}
  x &= 3 + 2s \\
  y &= -2 \\
  z &= 3 + 2s
\end{align*}
\]

We have

\[
\langle x - 1, y, z - 5 \rangle = t\langle -2, 2, -1 \rangle \quad \text{and} \quad \langle x - 3, y + 2, z - 3 \rangle = s\langle 2, 0, 2 \rangle.
\]

Since \(\langle -2, 2, -1 \rangle \neq c\langle 2, 0, 2 \rangle\), lines are not \(\parallel\).

Do lines intersect? Equate \(x\)'s:

\[1 - 2t = 3 + 2s \implies 2t = -2s - 2.\]

Equate \(y\)'s and substitute:

\[2t = -2 \implies -2 = -2s - 2 \implies s = 0 \quad \text{and} \quad t = -1.\]

Substitute into \(z\)'s:

\[5 + 1 \neq 3 + 0,\]

so the lines do not intersect, and thus we have skew lines.
Planes in \(\mathbb{R}^3\)

A plane in space is determined by a vector \(\mathbf{a} = \langle a_1, a_2, a_3 \rangle\) that is normal to the plane and a point \(P_1(x_1, y_1, z_1)\) lying in the plane.

A vector is normal to a plane if it is orthogonal to every vector in the plane.

1. Linear equation:

\[
\mathbf{a} \cdot \overrightarrow{P_1P} = \langle a_1, a_2, a_3 \rangle \cdot (x - x_1, y - y_1, z - z_1) = \left[ a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) \right] = 0
\]

Simplifying, every equation of the form

\[
ax + by + cz + d = 0
\]

is the equation of a plane with normal vector \(\langle a, b, c \rangle \neq \mathbf{0}\).

2. Vector equation:

\[
\mathbf{a} \cdot \overrightarrow{P_1P} = 0
\]

**Definition.** Two planes are parallel when their normal vectors are parallel, and orthogonal when their normal vectors are orthogonal.
**Example.** Three noncollinear points determine a plane. Find the plane containing \( P(-1, 3, 0) \), \( Q(3, 2, 4) \), and \( R(1, -1, 5) \).

We first find two vectors in the plane by using the points given.

\[
\overrightarrow{PQ} = \langle 4, -1, 4 \rangle \quad \text{and} \quad \overrightarrow{QR} = \langle -2, -3, 1 \rangle.
\]

We then find a normal vector \( \mathbf{n} \) by taking their cross product.

\[
\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
4 & -1 & 4 \\
-2 & -3 & 1
\end{vmatrix} = \langle 11, -12, -14 \rangle.
\]

Therefore, the equation is

\[
11(x + 1) - 12(y - 3) - 14(z - 0) = 0
\]

or

\[
11x - 12y - 14z + 47 = 0.
\]

Intercepts are

\[
x = -\frac{47}{11}, \quad y = \frac{47}{12}, \quad z = \frac{47}{14}.
\]
PROBLEM (Page 740 #24). Find the plane containing the point \((3, 0, -1)\) and \(\perp\) to the planes \(x + 2y - z = 2\) and \(2x - z = 1\).

Normal vectors to the two given planes are \(\mathbf{n}_1 = \langle 1, 2, -1 \rangle\) and \(\mathbf{n}_2 = \langle 2, 0, -1 \rangle\). Then a normal vector to the sought plane is

\[
\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & 0 & -1 \end{vmatrix} = \langle -2, -1, -4 \rangle.
\]

Then the equation is

\[-2(x - 3) - y - 4(z + 1) = 0\]

or

\[2x + y + 4z = 2.\]
Distance from a point $P_0$ to a plane $ax + by + cz + d = 0$

We measure the distance orthogonal to the plane.

(1) Choose any point $P_1$ in the plane.
(2) $a = \langle a, b, c \rangle$ is a normal vector to the plane.
(3) The distance $d$ is then

$$d = |\text{comp}_a \overrightarrow{P_1P_0}|.$$

**Problem** (Page 741 #40). Find the distance from $(1, 3, 0)$ to $3x + y - 5z = 2$.

$P_0 = (1, 3, 0)$, a normal vector is $\langle 3, 1, -5 \rangle$, and we choose $P_1 = (0, 2, 0)$ since it satisfies the equation. Then $\overrightarrow{P_1P_0} = \langle 1, 1, 0 \rangle$, and

$$d = |\text{comp}_{\langle 3,1,-5 \rangle} \langle 1, 1, 0 \rangle| = \left| \frac{\langle 1, 1, 0 \rangle \cdot \langle 3, 1, -5 \rangle}{\| \langle 3, 1, -5 \rangle \|} \right| = \left| \frac{4}{\sqrt{35}} \right| = \frac{4}{\sqrt{35}}$$

**Note.** You can see the derivation of the distance formula on page 739 of the text.
**Problem** (Page 740 #36 — Finding the intersection of 2 planes). Find the intersection of $3x + y - z = 2$ and $2x - 3y + z = -1$.

(1) Possibilities are that the intersection is a line, a plane, or does not exist (the planes are parallel.

Since normal vectors to the planes, $\langle 3, 1, -1 \rangle$ and $\langle 2, -3, 1 \rangle$, are not parallel, the intersection here is a line.

(2) Solve each equation for the same variable $- z$ seems easiest here.

$$z = 3x + y - 2 \quad \text{and} \quad z = -2x + 3y - 1$$

(3) Set these equal to each other.

$$3x + y - 2 = -2x + 3y - 1$$

(4) Solve for a second variable $- we choose $y$.

$$2y = 5x - 1 \implies y = \frac{5}{2}x - \frac{1}{2}$$

(5) Substitute this into either first equation.

$$z = 3x + \left(\frac{5}{2}x - \frac{1}{2}\right) - 2 \implies z = \frac{11}{2}x - \frac{5}{2}$$

or

$$z = -2x + 3\left(\frac{5}{2}x - \frac{1}{2}\right) - 1 \implies z = \frac{11}{2}x - \frac{5}{2}$$

(6) Set the third variable equal to a parameter $t$.

$$x = t$$

(7) Then parametric equations for the line are

$$\begin{align*}
  x &= t \\
  y &= \frac{5}{2}t - \frac{1}{2} \\
  z &= \frac{11}{2}t - \frac{5}{2}
\end{align*}$$

**Note.** Compare this problem to Example 5.8 on page 738.
MAPLE. See Lines and Planes(10.5).mw or Lines and Planes(10.5).pdf
Also see Matching Lines.

6. Sufaces in Space

MAPLE. See surfaces(10.6).mw or surfaces(10.6).pdf
Also see Drawing Surfaces, Drawing 3D Surfaces, and Matching Quadric Surfaces.