CHAPTER 1

Introduction

1. Background

Models of physical situations from Calculus

(1) Rate of change:

"A swimming pool is emptying at a constant rate of 90 gal/min."

With V = volume in gallons and t = time in minutes,

$$\frac{dV}{dt} = -90 \Longrightarrow$$
$$V = -90t + C$$

What is C?

C is the initial volume in the pool.

(2) Proportionality:

"The growth of a bacterial culture is proportional to the population present."

With w = the weight in grams of a bacterial culture and t = time in days,

$$\frac{dw}{dt} \propto w \Longrightarrow \frac{dw}{dt} = kw, \quad k > 0 \Longrightarrow$$
$$\frac{1}{w}\,dw = k\,dt \Longrightarrow \int \frac{1}{w}\,dw = \int k\,dt + C_1 \Longrightarrow$$
$$\ln w = kt + C_1 \Longrightarrow e^{\ln w} = e^{kt+C_1} \Longrightarrow$$
$$w = e^{kt}e^{C_1} \Longrightarrow w = Ce^{kt}$$

where $C > 0$.

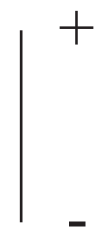
(3) Newton's 2nd Law of Motion:

A vector equation:

$$\mathbf{F}_{\text{sum}} = m\mathbf{a} \Longrightarrow$$

$$\mathbf{F}_{\text{sum}} = m\frac{d\mathbf{v}}{dt} \Longrightarrow \mathbf{F}_{\text{sum}} = m\frac{d^2\mathbf{s}}{dt^2}.$$

Note that $|\mathbf{F}| \propto |\mathbf{a}|$.



For a freely falling object, $F = ma = -mg$ where $a = -g < 0$. So, using h for height instead of s for position,

$$\frac{dv}{dt} = \frac{d^2h}{dt^2} = -g \Longrightarrow \underbrace{\int \frac{d^2h}{dt^2}\,dt = \int (-g)\,dt + v_0}_{\text{always add constant at point of integration}} \Longrightarrow$$

$$v = \frac{dh}{dt} = -gt + v_0 \Longrightarrow$$

$$\int \frac{dh}{dt}\,dt = \int (-gt + v_0)\,dt + s_0 \Longrightarrow$$

$$h = -\frac{gt^2}{2} + v_0 t + s_0$$

where the constants v_0 and s_0 are the initial veclocity and position of the body, respectively.

Definition. A differential equation is an equation that involves one or more derivatives of some unknown function or functions.

DEFINITION. An ordinary differential equation (ODE) is an equation that involves a single independent variable, one or more variables that depend only on the independent variable, and ordinary derivatives of one or more of these dependent variables.

DEFINITION. A patial differential equation (PDE) is an equation that involves two or more independent variables, one or more variables that depend only on the independent variables, and partial derivatives of one or more of these dependent variables.

DEFINITION. The order of an ODE is said to be n if the order of the highest derivative appearing in the equation is n.

DEFINITION. A parameter is a quantity that does not change as the independent variable changes (for example, k and m in the preceding models). However, they may change as a situation changes or different equipment is used in an experiment (for instance, using objects of varying mass).

DEFINITION. An ordinary differential equation is called a linear differential equation if it has the format

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$

where $a_n(x), a_{n-1}(x), \dots, a_0(x)$, and $F(x)$ depend only on the independent variable x.

EXAMPLE.

(1) $3x\dfrac{d^2y}{dx^2} + \sin x\dfrac{dy}{dx} + (\cos x)y = x^2 \sin x$

is linear.

(2) $y\dfrac{dy}{dx} + (\sin x)y^3 + \dfrac{3}{y^2} = e^x + 1$

is not linear

2. Solutions and Initial Value Problems

DEFINITION. An nth-order ordinary differential equation is an equality relating the independent variable to the nth deriviative (and usually lower order derivatives as well) of the dependent variable.

EXAMPLE.

$$\frac{d^3y}{dx^3} + x^3\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = x^2$$

is a third-order ODE with independent variable x and dependent variable y.

Thus a general form for an nth-order ODE can be expressed as

$$(*) \qquad F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

where F is a function that depends on $x, y, \dfrac{dy}{dx}, \dots, \dfrac{d^ny}{dx^n}$. We assume the equation holds for all x in an open interval I ($a < x < b$, where a and/or b could be infinite. We can also isolate the highest order derivative and write the equation as

$$(**) \qquad \frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right).$$

DEFINITION (1 — Explicit Solution). A function $y = \phi(x)$ (or just $y(x)$) that when substituted for y in equation $(*)$ or $(**)$ satisfies the equation for all x in the interval I is called an <u>explicit solution</u> to the equation on I.

EXAMPLE.

(1) Is $y = \dfrac{1}{25}e^{3x}$ an explicit solution of $\dfrac{d^2y}{dx^2} + 16y = e^{3x}$?

SOLUTION.

$$\frac{dy}{dx} = \frac{3}{25}e^{3x}, \qquad \frac{d^2y}{dx^2} = \frac{9}{25}e^{3x}.$$

$$\frac{d^2y}{dx^2} + 16y = \frac{9}{25}e^{3x} + \frac{16}{25}e^{3x} = e^{3x}.$$

Thus y is an explicit solution of the equation. □

(2) Is $y = e^{2x}$ an explicit solution of $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$?

SOLUTION.

$$\frac{dy}{dx} = 2e^{2x}, \qquad \frac{d^2y}{dx^2} = 4e^{2x}.$$

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 8e^{2x} - 14e^{2x} + 3e^{2x} = -3e^{2x} \neq 0.$$

Thus y is not an explicit solution of the equation. □

(3) For $t > 0$, is $x = t\tan(\ln t)$ an explicit solution of $\frac{dx}{dt} = \frac{t^2 + tx + x^2}{t^2}$?

SOLUTION.

$$\frac{dx}{dt} = \tan(\ln t) + t\sec^2(\ln t) \cdot \frac{1}{t} = \tan(\ln t) + \sec^2(\ln t)$$

and

$$\frac{t^2 + tx + x^2}{t^2} = 1 + \frac{1}{t}x + \frac{1}{t^2}x^2 = 1 + \tan(\ln t) + \tan^2(\ln t) = \tan(\ln t) + \sec^2(\ln t).$$

Since both sides evaluate to the same expression, x is an explicit solution of the equation. □

EXAMPLE. Assuming $x^2 + y^2 = 4$ implicitly defines y as a function of x, does it implicity define one or more solutions to the equation $\frac{dy}{dx} = -\frac{x}{y}$?

SOLUTION. Differentiating $x^2 + y^2 = 4$ implicitly,

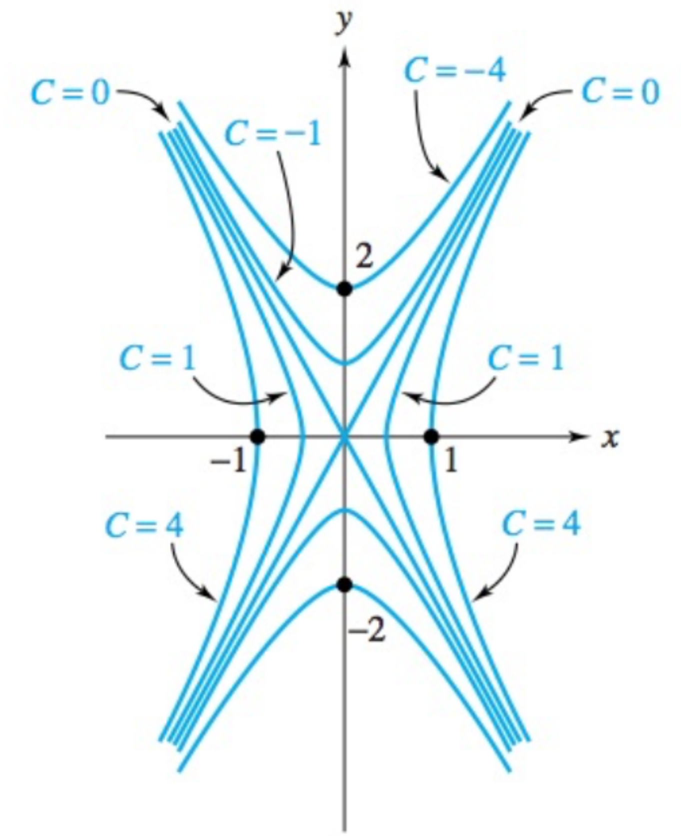
$$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow 2y\frac{dy}{dx} = -2x \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Thus $x^2 + y^2 = 4$ defines one or more solutions to the equation $\frac{dy}{dx} = -\frac{x}{y}$.

In fact, $y = \pm\sqrt{4 - x^2}$ are two explict solutions. □

DEFINITION (2 — Implicit Solution). A relation $G(x, y) = 0$ is said to be an implicit solution to equations $(*)$ and $(**)$ on the interval I if it defines one or more explicit Solutions on I.

EXAMPLE. Show that for every constant C the relation $4x^2 - y^2 = C$ is an implcit solution the DE $y\dfrac{dy}{dx} - 4x = 0$. The solution curves for $C = 0, \pm 1, \pm 4$ (a one-parameter family of solutions) is shown below.



The curves are hyperbolas with common asymptotes $y = \pm 2x$. For $C = 0$, $y = \pm 2x$ are explicit solutions.

SOLUTION.

Implicitly differentiating $4x^2 - y^2 = C$ with respect to x, we get

$$8x - 2y\frac{dy}{dx} = 0 \Longrightarrow y\frac{dy}{dx} - 4x = 0.$$

Thus $4x^2 - y^2 = C$ is an implicit solution to the DE. □

PROBLEM (Page 14 #16). Verify that $x^2 + cy^2 = 1$, where c is an arbitrary nonzero constant, is a one-parameter family of solutions to

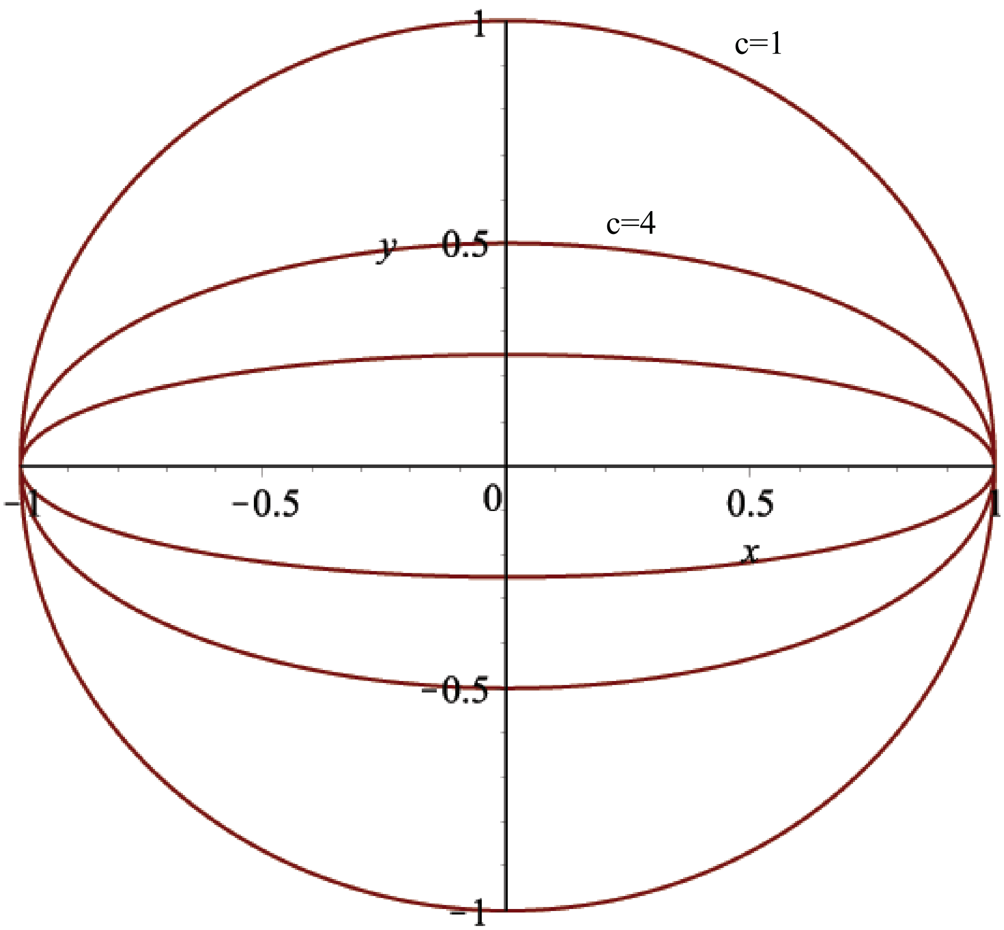
$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

and graph several solution curves using the same coordinate axes.

SOLUTION. Differentiating $x^2 + cy^2 = 1$ implicitly with respect to x,

$$2x + 2cy\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{cy} = -\frac{xy}{cy^2} = -\frac{xy}{1 - x^2} \Longrightarrow \frac{dy}{dx} = \frac{xy}{x^2 - 1}.$$

Solution curves for $c = 1, 4, 9$:



□

DEFINITION (3 — Initial Value Problem). By an <u>initial value problem</u> for an nth-order DE

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0,$$

we mean: Find a solution to the DE on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_1, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

NOTE.

(1) For a first-order equation, the initial condition (IC) is simply

$$y(x_0) = y_0.$$

(2) For a second-order equation, the IC are

$$y(x_0) = y_0, \qquad \frac{dy}{dx}(x_0) = y_1.$$

(3) In mechanics, where t, representing time, is the independent variable instead of x, and y represents position, if t_0 is the starting time, $y(t_0) = y_0$ is the initial position of an object and $y'(t_0) = y_1$ is its initial velocity.

PROBLEM (Page 14 #22a). Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear DE

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the following IC are satisfied:

$$y(0) = 2, \qquad y'(0) = 1.$$

SOLUTION. For $y = c_1e^x + c_2e^{-2x}$,

$$\frac{dy}{dx} = c_1e^x - 2c_2e^{-2x} \quad \text{and} \quad \frac{d^2y}{dx^2} = c_1e^x + 4c_2e^{-2x},$$

so

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = c_1e^x + 4c_2e^{-2x} + c_1e^x - 2c_2e^{-2x} - 2(c_1e^x + c_2e^{-2x}) = 0$$

and thus $\phi(x)$ is a solution to the DE. From the initial conditions,

$$(*) \quad y(0) = c_1 + c_2 = 2$$
$$(**) \quad y'(0) = c_1 - 2c_2 = 1.$$

Subtracting $(**)$ from $(*)$, we get $3c_2 = 1 \Longrightarrow c_2 = \frac{1}{3} \Longrightarrow c_1 = \frac{5}{3}$. □

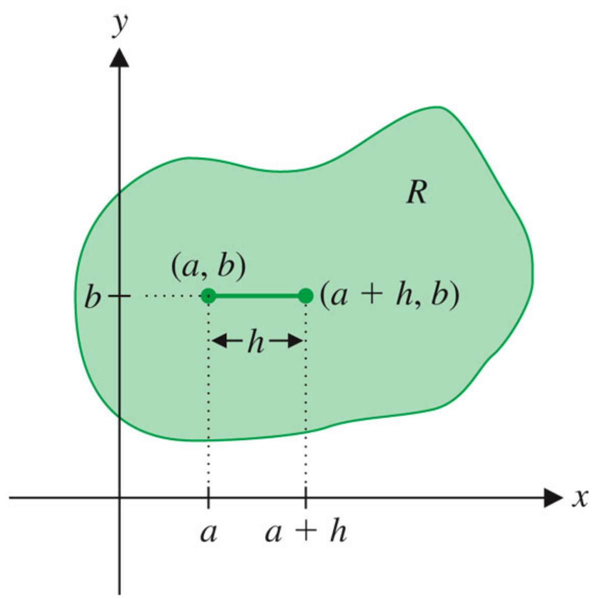
For the next important theorem, we need to introduce the topic of partial derivatives. But first, we look at a Maple worksheet regarding graphs of functions of two variables.

MAPLE. See function 2 variable.mw or function 2 variable.pdf

Partial Derivatives

First-order partial derivatives: Consider a function $f(x, y)$ defined on a region $R \in \mathbb{R}^2$. Let (a, b) be an interior point of R. The average rate of change as you move horizontally from (a, b) to $(a + h, b)$ is

$$\frac{f(a+h, b) - f(a, b)}{h}.$$



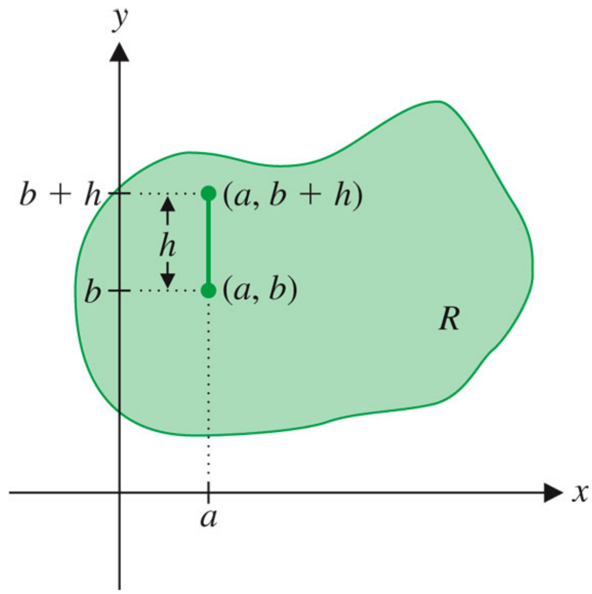
The instantaneous rate of change in the x-direction at (a, b) is

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h},$$

the partial derivative of f with respect to x.

The average rate of change as you move vertically from (a, b) to $(a, b+h)$ is

$$\frac{f(a, b+h) - f(a, b)}{h}.$$



The instantaneous rate of change in the y-direction at (a, b) is

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

the partial derivative of f with respect to y.

EXAMPLE. Let $f(x, y) = x^2 y^2$.

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{(x+h)^2 y^2 - x^2 y^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 y^2 + 2xhy^2 + h^2 y^2 - x^2 y^2}{h} = \lim_{h \to 0} \frac{2xhy^2 + h^2 y^2}{h} \\ &= \lim_{h \to 0} (2xy^2 + hy^2) = 2xy^2.\end{aligned}$$

Basically, hold y constant and take the derivative with respect to x. We do similarly for $\frac{\partial f}{\partial y}(x, y)$. Then, for example,

$$\frac{\partial f}{\partial x}(3, 2) = 2 \cdot 3 \cdot 2^2 = 24.$$

NOTATION.

$$\underbrace{\frac{\partial f}{\partial x}(x,y)}_{\text{traditional notation}} = \underbrace{f_x(x,y)}_{\text{modern notation}} = \underbrace{\frac{\partial}{\partial x}\big[f(x,y)\big]}_{\text{partial differential operator}}$$

$$\overbrace{\frac{\partial f}{\partial y}}(x,y) = \overbrace{f_y(x,y)} = \overbrace{\frac{\partial}{\partial y}}\big[f(x,y)\big]$$

EXAMPLE.

$$\frac{\partial}{\partial x}(xe^{\sqrt{xy}}) = \frac{\partial}{\partial x}(xe^{(xy)^{1/2}}) = e^{(xy)^{1/2}} + xe^{(xy)^{1/2}}\left(\frac{1}{2}\right)(xy)^{-1/2}(y) = \left(\frac{1}{2}\right)e^{\sqrt{xy}}(2+\sqrt{xy}).$$

NOTE.

$$x \underset{\boxed{y}}{\longrightarrow} xy \underset{\boxed{\frac{1}{2}z^{-1/2}}}{\longrightarrow} (xy)^{1/2} \underset{\boxed{e^s}}{\longrightarrow} e^{(xy)^{1/2}}$$

$$xy = z, \quad (xy)^{1/2} = z^{1/2} = s, \quad e^{(xy)^{1/2}} = e^{z^{1/2}} = e^s$$

EXAMPLE. $f(x,y) = x\ln(y\cos x)$. Find $\dfrac{\partial f}{\partial x}\left(\dfrac{\pi}{3},1\right)$.

$$\frac{\partial f}{\partial x}(x,y) = \ln(y\cos x) + x\,\frac{1}{y\cos x}\,(-y\sin x) = \ln(y\cos x) - x\tan x$$

NOTE.

$$x \underset{\boxed{-\sin x}}{\longrightarrow} \cos x \underset{\boxed{y}}{\longrightarrow} y\cos x \underset{\boxed{\frac{1}{z}}}{\longrightarrow} \ln(y\cos x)$$

$$\cos x = s, \quad y\cos x = ys = z, \quad \ln(y\cos x) = \ln(ys) = \ln z$$

Thus

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{3},1\right) = \ln\frac{1}{2} - \frac{\pi}{3}\sqrt{3} = -\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right).$$

MAPLE. See partderiv.mw or partderiv.pdf

Given an initial value problem (IVP), we certainly hope there is a solution and, furthermore, that there is only one solution. The following theorem gives us conditions that guarantee this.

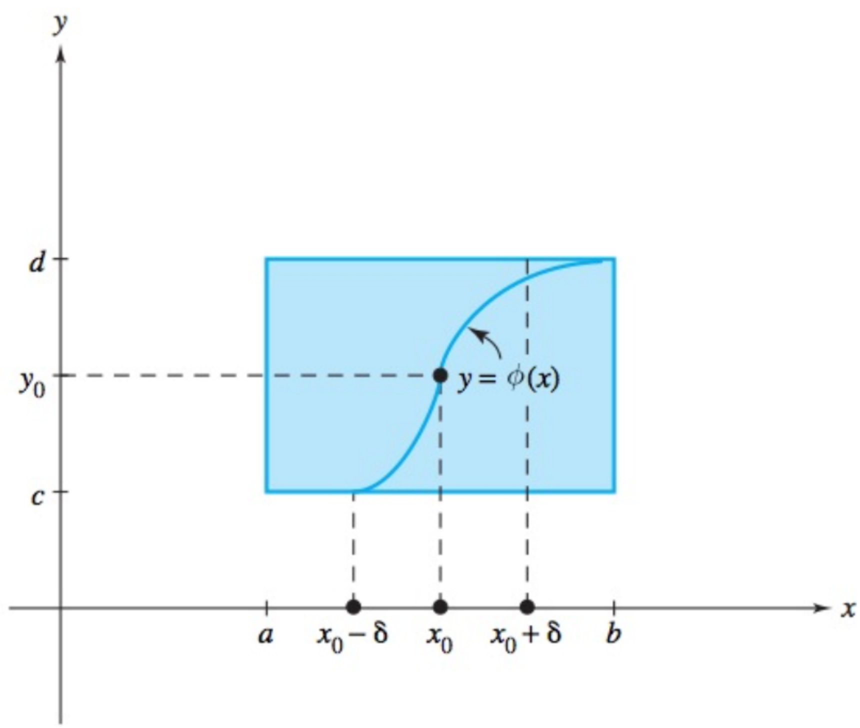
THEOREM (1 – Existence and Uniqueness of Solution). *Consider the IVP*

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

If f and $\dfrac{\partial f}{\partial y}$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the IVP has a unique solution $y = \phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.



Layout for the existence–uniqueness theorem

PROBLEM (Page 14 # 26). Does Theorem 1 imply that

$$\frac{dx}{dt} + \cos x = \sin t, \qquad x(\pi) = 0$$

has a unique solution.

SOLUTION.

Changing the equation to $\frac{dx}{dt} = \sin t - \cos x$ giving

$$f(t, x) = \sin t - \cos x \implies \frac{\partial f}{\partial x} = \sin x$$

and both these functions are continuous for all t and x, by Theorem 1 the above IVP does have a unique solution for all t. □

PROBLEM (Page 14 # 28). Does Theorem 1 imply that

$$\frac{dy}{dx} = 3x - \sqrt[3]{y-1}, \qquad y(2) = 1$$

has a unique solution.

SOLUTION.

We have

$$f(x, y) = 3x - (y-1)^{1/3} \implies$$
$$\frac{\partial f}{\partial y} = -\frac{1}{3}(y-1)^{-2/3} = -\frac{1}{\sqrt[3]{(y-1)^2}}.$$

Since $\frac{\partial f}{\partial y}$ is undefined at the point $(2, 1)$, it cannot be continuous in any rectangle containing that point, so Theorem 1 does not guarantee a unique solution. □

NOTE. In cases where Theorem 1 does not guarantee a unique solution, there still may be one or more solutions.

3. Direction Fields

MAPLE. See direction fields.mw or direction fields.pdf

4. The Approximation Method of Euler

MAPLE. See euler example.mw or euler example.pdf.

Euler's method finds a polygonal path to approximate the solution of an IVP

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

by computing a sequence of points that are then joined by lines as follows:

(1) First decide on a suitable step size h. If we wish a solution for $x_0 \leq x \leq b$ with N steps, we let $h = \dfrac{b - x_0}{N}$.

(2) Start at (x_0, y_0).

(3) Repeat the following algorithm until a desired stopping point on the x-axis is reached:

Determine (x_{n+1}, y_{n+1}) from (x_n, y_n) by calculating

(a) the slope $f(x_n, y_n)$ from $\dfrac{dy}{dx} = f(x, y)$

(b) and then the coordinates from the iteration formulas

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

EXAMPLE. $\frac{dy}{dx} = x - y, \quad y(0) = 0, \quad 0 \le x \le .5, \quad \text{five steps}$

Solution

$$h = \frac{.5 - 0}{5} = \frac{.5}{5} = .1 \text{ and } y_{n+1} = y_n + .1(x_n - y_n)$$

$x_0 = 0$

$y_0 = 0$

——————————————————(0, 0)

$x_1 = 0 + .1 = .1$

$y_1 = 0 + .1(0 - 0) = 0$

——————————————————(.1, 0)

$x_2 = .1 + .1 = .2$

$y_2 = 0 + .1(.1 - 0) = .01$

——————————————————(.2, .01)

$x_3 = .2 + .1 = .3$

$y_3 = .01 + .1(.2 - .01) = .029$

——————————————————(.3, .029)

$x_4 = .3 + .1 = .4$

$y_4 = .029 + .1(.3 - .029) = .0561$

——————————————————(.4, .0561)

$x_5 = .4 + .1 = .5$

$y_5 = .0561 + .1(.4 - .0561) = .09049$

——————————————————(.5, .09049)

MAPLE. See euler-maple.mw or euler-maple.pdf.