CHAPTER 4

Linear Programming

1. Graphing Linear Inequalities

**Notation.**

- $x \leq y$ means $x$ is less than or equal to $y$.
- $x \geq y$ means $x$ is greater than or equal to $y$.
- $x < y$ means $x$ is less than $y$.
- $x > y$ means $x$ is greater than $y$.

The last two inequalities are called **strict inequalities**. Our focus will be on the **nonstrict inequalities**.

**Algebra of Inequalities**

Suppose

$$x + 3 < 8.$$ 

Addition works like for equations:

$$x + 6 < 11 \quad (\text{added 3 to each side}).$$

Subtraction works like for equations:

$$x + 2 < 7 \quad (\text{subtracted 4 from each side}).$$

Multiplication and division by positive numbers work like for equations:

$$2x + 12 < 22 \implies x + 6 < 11$$

(each side is divided by 2 or multiplied by $\frac{1}{2}$).
Multiplication and division by negative numbers changes the direction of the inequality sign:

\[ 2x + 12 < 22 \implies -x - 6 > -11 \]

(each side is divided by -2 or multiplied by \(-\frac{1}{2}\)).

**Example.** For \(3x - 4y\) and 24 there are 3 possibilities:

\[
\begin{align*}
3x - 4y &= 24 \\
-4y &= -3x + 24 \\
y &= \frac{3}{4}x - 6
\end{align*}
\]

\[
\begin{align*}
3x - 4y &< 24 \\
-4y &< -3x + 24 \\
y &> \frac{3}{4}x - 6
\end{align*}
\]

\[
\begin{align*}
3x - 4y &> 24 \\
-4y &> -3x + 24 \\
y &< \frac{3}{4}x - 6
\end{align*}
\]

The three solution sets above are disjoint (do not intersect or overlap), and their graphs fill up the plane. We are familiar with the graph of the linear equation. The graph of one inequality is all the points on one side of the line, the graph of the other all the points on the other side of the line.

To determine which side for an inequality, choose a test point not on the line (such as \((0, 0)\) if the line does not pass through the origin). Substitute this point into the linear inequality. For a true statement, the solution region is the side of the line that the test point is on; for a false statement, it is the other side.
\[ 3x - 4y < 24 \]
\[ 0 - 0 < 24 \text{ (true)} \]

\[ y > \frac{3}{4}x - 6 \]
\[ 0 > 0 - 6 \text{ (true)} \]

\[ 3x - 4y > 24 \]
\[ 0 - 0 > 24 \text{ (false)} \]

\[ y < \frac{3}{4}x - 6 \]
\[ 0 < 0 - 6 \text{ (false)} \]

Statement is true, so take side of line \((0, 0)\) is on.

Statement is false, so take side opposite \((0, 0)\).

The dashed lined indicates that points on the line are not part of the solution. For \(3x - 4y \leq 24\) or \(3x - 4y \geq 24\), we would use a solid line to indicate that the boundary line is part of the solution.

**Note.**

(1) For \(y \geq mx + b\), the solution region is the line with the points above it.

(2) For \(y \leq mx + b\), the solution region is the line with the points below it.
Example. \(2x \geq 7\) or \(x \geq \frac{7}{2}\) or \(x + 0y \geq \frac{7}{2}\).

\[
6 + 0(2) \geq \frac{7}{2} \quad \text{(true)}
\]

Thus we take points to the right of the line and we use a solid line because of \(\geq\).
EXAMPLE. A system of inequalities:

\[
\begin{align*}
(1) & \quad 2x + 5y \leq 20 \\
(2) & \quad x - 5y \geq -5
\end{align*}
\]

Intercepts for (1) are 10 and 4. Intercepts for (2) are $-5$ and 1.

The test point of $(0, 0)$ makes each statement true, so take the points below each line. The solution of the system is the intersection of the individual solutions (in blue).

It will be important for linear programming to find the points of intersection of our lines. Here, it appears to be $(5, 2)$. But, as for any graphic solution, we need to check this solution in both equations (1) and (2).

\[
\begin{align*}
2(5) + 5(2) &= 20 \quad \text{(true)} \\
5 - 5(2) &= -5 \quad \text{(true)}
\end{align*}
\]

One can also use the algebraic methods studied earlier to find points of intersection.
For the TI, rewrite in slope-intercept form

\[
\begin{align*}
2x + 5y & \leq 20 \\
 x - 5y & \geq -5
\end{align*}
\] \Rightarrow \begin{align*}
5y & \leq -2x + 20 \\
-5y & \geq -x - 5
\end{align*} \Rightarrow \begin{align*}
y & \leq -\frac{2}{5}x + 4 \\
y & \leq \frac{1}{5}x + 1
\end{align*}

and use the Technology Tip on page 180 of the text.

**Example.**

\[
\begin{align*}
(1) \quad 3x + y & \leq 21 \\
(2) \quad x + y & \leq 9 \\
(3) \quad x + 3y & \leq 21 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

The last two inequalities force our region to be limited to the first quadrant. The intercepts for (1) are 7 and 21, for (2) 9 and 9, and for (3) 21 and 7. (0, 0) makes each of (1)–(3) true, so the solution regions are on the same side of the lines as (0, 0).
Numbering the inequalities and lines helps us to find intersection points or corners of our solution region. Lines (1) and (2) intersect at the corner \((6, 3)\), and lines (2) and (3) intersect at \((3, 6)\). Other corners are \((0, 0)\), \((7, 0)\), and \((0, 7)\).

This solution region is bounded since it can be contained in a circle.

**Example.**

\[
\begin{align*}
(1) \quad 3x + y & \geq 24 \\
(2) \quad x + y & \geq 16 \\
(3) \quad x + 3y & \geq 30 \\
\quad x & \geq 0 \\
\quad y & \geq 0
\end{align*}
\]

The intercepts for (1) are 8 and 24, for (2) 16 and 16, and for (3) 30 and 10. \((0, 0)\) makes each of (1)–(3) false, so the solution regions are on the side of the lines opposite \((0, 0)\).

Corners of the solution region include the \(x\)-intercept at 30 and the \(y\)-intercept at 24, along with \((4, 12)\), the intersection of lines (1) and (2), and \((9, 7)\), the
intersection of lines (2) and (3). This is an unbounded region since it cannot be enclosed in a circle.

**Example.**

\[
\begin{align*}
(1) & \quad 2x + 3y \geq 24 \\
(2) & \quad x + 3y \leq 15 \\
(3) & \quad y \geq 4
\end{align*}
\]

The intercepts for (1) are 12 and 8, for (2) 15 and 5, and for (3) \( y = 4 \). \((0, 0)\) makes (2) true and (1) and (3) false, so solution region (1) is the same side of the line as \((0, 0)\) and the solution regions (2) and (3) are on the side of the lines opposite \((0, 0)\).

For this example, there is no solution since no region satisfies all three inequalities.
**Problem** (Page 184 #30).

\[ x = \# \text{ of cartons of medium peaches (60 peaches per carton)} \]

\[ y = \# \text{ of cartons of small peaches (70 peaches per carton)} \]

\[
\begin{align*}
(1) & \quad 9x + 10y \leq 100 \quad \text{(budget)} \\
(2) & \quad 60x + 70y \geq 420 \quad \text{(minimum)} \\
(2) & \quad 60x + 70y \leq 630 \quad \text{(maximum)}
\end{align*}
\]

The intercepts for (1) are \(11\frac{1}{9}\) and 10, for (2) 7 and 6, and for (3) \(10\frac{1}{2}\) and 9. (0, 0) makes (1) and (3) true and (2) false, so solution regions (1) and (3) are on the same side of the line as (0, 0) and the solution region (2) is on the side of the line opposite (0, 0).

The feasible region consists of all points with both entries integers in the first quadrant on or between lines (2) and (3). All possible such choices are circled above.
2. Solving Linear Programming Problems Graphically

Example.

Manufacturing: resource allocation. An electronics firm manufactures two types of personal computers, a standard model and a portable model. The production of a standard computer requires a capital expenditure of $400 and 40 hours of labor. The production of a portable computer requires a capital expenditure of $250 and 30 hours of labor. The firm has $20,000 capital and 2,160 labor-hours available for production of standard and portable computers.

(A) What is the maximum number of computers the company is capable of producing?

\[ M = \# \text{ of computers to be made.} \]
\[ x = \# \text{ of standard computers to be made.} \]
\[ y = \# \text{ of portable computers to be made.} \]

Objective:

Maximize \[ M = x + y \] \hspace{1em} \text{(objective function)}

\( M \) is the objective variable, a variable we try to maximize or minimize. \( x \) and \( y \) are decision variables – we try various values to reach a decision.

subject to \[
\begin{align*}
(1) & \quad 400x + 250y \leq 20000 \\
(2) & \quad 40x + 30y \leq 2160 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

These are constraints.

The above is a linear programming problem. The values of \( x \) and \( y \) that optimize (maximize or minimize) the value of the objective function are called the optimal solution. A linear programming problem with the additional constraint that \( x \) and \( y \) are integers is called an integer programming problem.
To solve, we first find the feasible region, the solution region of the system of constraints. The points within the feasible region are feasible points, one or more of which will optimize the objective function.

Intercepts for (1) are 50 and 80, for (2) are 54 and 72. We will use divisions of 5 on the $x$-axis and 8 on the $y$-axis.

We find the point where the two lines intersect.

\[
\begin{align*}
400x + 250y &= 20000 \\
-400x - 300y &= -21600 \\
-50y &= -1600 \\
y &= 32 \\
40x + 30(32) &= 2160 \\
40x + 960 &= 2160 \\
40x &= 1200 \\
x &= 30
\end{align*}
\]
Thus the corners of the feasible region are (0, 0), (50, 0), (0, 72), and (30, 32).

We now look to the objective function $M = x + y$ and graph it on the feasible region for the cases where $M = 20, 30, 40, 50, \text{ and } 60$.

All of these lines are parallel with smaller $M$’s giving lines closer to the origin and larger $M$’s lines further from the origin. Think of these iso-$M$ lines as a rolling pin. Notice the rolling pin must leave the feasible region at one of its corner points, in this case (0, 0) for a minimum value and (0, 72) for a maximum value.

**Theorem** (Fundamental Theorem of Linear Programming).

(1) If the solution to a linear programming problem exists, it will occur at a corner point.

(2) If two adjacent corner points are optimal solutions, then all points on the line segment between them are also optimal solutions.

(3) Linear programming problems with bounded feasible regions will always have optimal solutions (both a maximum and a minimum).

(4) Linear programming problems with unbounded feasible regions may or may not have optimal solutions.
Thus, to find the solution to a linear programming problem, we need only check the value of the objective variable at the corner points of the feasible region.

<table>
<thead>
<tr>
<th>CP</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(50, 0)</td>
<td>50</td>
</tr>
<tr>
<td>(0, 72)</td>
<td>72</td>
</tr>
<tr>
<td>(30, 32)</td>
<td>62</td>
</tr>
</tbody>
</table>

Thus the maximum number of computers that can be produced with these constraints is 72, achieved by making 72 portables.

(B) If each standard computer contributes a profit of $320 and each portable model contributes a profit of $220, how much profit will the company make by producing the maximum number of computers determined in part (A)? Is this the maximum profit? If not, what is the maximum profit?

The profit in part (A) is $72($220) = $15840. Now let

$P = \text{profit},$

and change the objective to

**Maximize** $P = 320x + 220y$

Since we only changed the objective, and not the constraints, the feasible region and its corner points remain the same. The diagram below has **iso-**$P$** lines** for 4 values of $P$. 
We check the corner points:

<table>
<thead>
<tr>
<th>CP</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(50, 0)</td>
<td>16000</td>
</tr>
<tr>
<td>(0, 72)</td>
<td>15840</td>
</tr>
<tr>
<td>(30, 32)</td>
<td>16640</td>
</tr>
</tbody>
</table>

Thus the maximum profit is $16,640 and can be achieved by making 30 standard and 32 portable models. □

Graphical Methods for Solving Linear Programming Problems

(1) Graph the feasible region determined by the constrains.

(2) Find the corner points of the feasible region.

(3) Find the value of the objective function at each of the corner points.

(4) If the feasible region is bounded, the maximum or minimum value of the objective function will occur at one of the corner points.
(5) If the feasible region is an unbounded region in the first quadrant and the coefficients of the objective function are positive, then the objective function has a minimum value at a corner point. The objective function will not have a maximum value.

**Problem** (Page 199 #4).

Maximize $z = 9x + y$

subject to

\[
\begin{align*}
(1) & \quad 6x + y \leq 16 \\
(2) & \quad -2x + y \leq 0 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

Intercepts for (1) are $\frac{8}{3}$ and 16. (2) has only the origin as an intercept, but also pass through (1, 2). Since (0, 0) makes (1) a true statement, we take the points below line (1). Since (2) passes through the origin, we take (1, 0) as a test point, and this makes (2) true, so we take the points below (2).

Our feasible region has (0, 0), ($\frac{8}{3}$, 0), and (2, 4) as corner points.
Thus the optimal solution is \((\frac{8}{3}, 0)\) and the optimal value is 24.

**Problem** (Page 199 #12).

Minimize \(z = 3x + 5y\)

subject to \[
\begin{align*}
(1) & \quad 6x + y \geq 21 \\
(2) & \quad -2x + y \geq 1 \\
(3) & \quad x \geq 3 \\
& \quad y \geq 0
\end{align*}
\]

Intercepts for (1) are \(\frac{7}{2}\) and 21 and for (2) are \(-\frac{1}{2}\) and 1. Since the intercepts for (2) are so close together, we also use the point (1, 3) as a guide in drawing the line. Since \((0, 0)\) makes all of (1)–(3) false, we take the regions above (1) and (2) and to the right of (3).
The only corner point is (3, 7). The value of $z$ at (3, 7) is $9 + 35 = 44$, and by point (5) of the graphical method, this must be a minimum. Note that, with these constraints, $z$ has no maximum.

**PROBLEM** (Page 201 #28).

$x = \#$ of armchairs to produce.

$y = \#$ of benches to produce.

$P =$ profit.

Maximize $P = 181x + 57y$

subject to

\[
\begin{align*}
(1) \quad & 2x + y \leq 200 \quad \text{(cutting)} \\
(2) \quad & 4x + 3y \leq 480 \quad \text{(finishing)} \\
(3) \quad & x + y \leq 150 \quad \text{(packaging)} \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]

Intercepts for (1) are 100 and 200, for (2) are 120 and 160, and for (3) are 150 and 150. Since (0, 0) makes all of (1)–(3) true, we choose the regions below each line.
Corner points of the feasible region are \((0, 0), (100, 0), (60, 80)\) – the intersection of lines \((1)\) and \((2)\), \((30, 120)\) – the intersection of lines \((2)\) and \((3)\), and \((0, 150)\). Notice that \((50, 100)\) – the intersection of lines \((1)\) and \((2)\) – is not a corner point of the feasible region.

<table>
<thead>
<tr>
<th>CP</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>0</td>
</tr>
<tr>
<td>((100, 0))</td>
<td>18,100</td>
</tr>
<tr>
<td>((60, 80))</td>
<td>15,420</td>
</tr>
<tr>
<td>((30, 120))</td>
<td>12,270</td>
</tr>
<tr>
<td>((0, 150))</td>
<td>8,550</td>
</tr>
</tbody>
</table>

You can obtain a maximum profit of $18,100 by making 100 armchairs and 0 benches.

**Problem** (Page 202 #30).

\[ x = \# \text{ of 10-passenger vehicles.} \]

\[ y = \# \text{ of 57-passenger vehicles.} \]

\[ C = \text{cost.} \]

Minimize \[ C = 625x + 1995y \]

subject to \[
\begin{align*}
(1) & \quad 9x + 51y \geq 153 \quad \text{(students)} \\
(2) & \quad x + 6y \leq 24 \quad \text{(chaperones)} \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

Intercepts for \((1)\) are 17 and 3 and for \((2)\) are 24 and 4. Since \((0, 0)\) makes \((1)\) false and \((2)\) true, we take the points above line \((1)\) and below line \((2)\).
The four corner points of the feasible region are the four intercepts.

<table>
<thead>
<tr>
<th>CP</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3)</td>
<td>5,985</td>
</tr>
<tr>
<td>(0, 4)</td>
<td>7,980</td>
</tr>
<tr>
<td>(17, 0)</td>
<td>10,625</td>
</tr>
<tr>
<td>(24, 0)</td>
<td>15,000</td>
</tr>
</tbody>
</table>

The minimum cost of $5985 can be achieved by renting three 57-passenger vehicles and no 10-passenger vehicles.