Exact Equations

```
> restart:with(plots):with(DEtools):
```

We consider differential equations of the form

$$M(x,y) + \frac{N(x,y)\,dy}{dx} = 0\,.$$

Recall that the **total derivative** $dF(x,y)$ of a function F(x,y) of two variables is defined by

$$\frac{dF}{dx} = \frac{\partial}{\partial x} F(x,y) + \frac{\left(\frac{\partial}{\partial y} F(x,y)\right) dy}{dx}\,.$$

When the right-hand side of the above differential equation is a total derivative for some function F(x,y), the equation is an exact equation. In this case, since dF(x,y) = 0, we know that the solution of the differential equation is F(x,y) = C, for some constant C. Why?

Thus, if we recognize that a DE is exact, we can simply integrate to find its general solution.

We also know that an equation of the form

$$M(x,y) + \frac{N(x,y)\,dy}{dx} = 0$$

is exact if

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y).$$

Let's use MAPLE's computational power to test a DE for exactness using the command odeadvisor.

```
> M(x,y(x)):= cos(x)*cos(y(x))+2*x;
```

$$M(x, y(x)) := \cos(x)\cos(y(x)) + 2x$$

```
> N(x,y(x)):= -sin(x)*sin(y(x))-2*y(x);
```

$$N(x, y(x)) := -\sin(x)\sin(y(x)) - 2\,y(x)$$

```
> deq:=M(x,y(x))+N(x,y(x))*diff(y(x),x)=0;
```

$$deq := \cos(x)\cos(y(x)) + 2x + (-\sin(x)\sin(y(x)) - 2\,y(x))\left(\frac{d}{dx} y(x)\right) = 0$$

```
> odeadvisor(deq,[exact]);
```

$$[\_exact]$$

This differential equation is exact since the word exact appears in the list generated by odeadvisor. Otherwise, it is not. We can also use odeadvisor to check for other types. Let's find the solution using Maple's power.

```
> dsolve(deq,y(x));
```

$$\sin(x)\cos(y(x)) + x^2 - y(x)^2 + \_C1 = 0$$

Let's look at another example.

```
> M(x,y(x)):= 1/y(x);
```

$$M(x, y(x)) := \frac{1}{y(x)}$$

```
> N(x,y(x)):= -3+x/y(x)^2;
```

$$N(x, y(x)) := -3 + \frac{x}{y(x)^2}$$

```
> deq:=M(x,y(x))+N(x,y(x))*diff(y(x),x)=0;
```

$$deq := \frac{1}{y(x)} + \left(-3 + \frac{x}{y(x)^2}\right)\left(\frac{d}{dx}\, y(x)\right) = 0$$

```
> odeadvisor(deq,[exact]);
```

$$[NONE]$$

This DE is not exact.

Integral Curves and Contours of Exact Equations

```
> restart:with(plots):with(DEtools):
```

We will learn how to use *Maple* commands to plot ***integral curves*** and ***contours*** of exact first order differential equations which are expressed in the form

$$M(x, y) + \frac{N(x, y)\, dy}{dx} = 0\,.$$

We begin with the exact equation

$$x + \frac{y\, dy}{dx} = 0\,.$$

```
> M(x,y(x)):= x;
```

$$M(x, y(x)) := x$$

```
> N(x,y(x)):= y(x);
```

$$N(x, y(x)) := y(x)$$

```
> deq:=M(x,y(x))+N(x,y(x))*diff(y(x),x)=0;
```

$$deq := x + y(x)\left(\frac{d}{dx}\, y(x)\right) = 0$$

```
> odeadvisor(deq,[exact]);
```

$$[\_exact]$$

We see this DE is exact. Let's get the solution.

```
> soln:=dsolve(deq,y(x),implicit);
```

$$soln := x^2 + y(x)^2 - \_C1 = 0$$

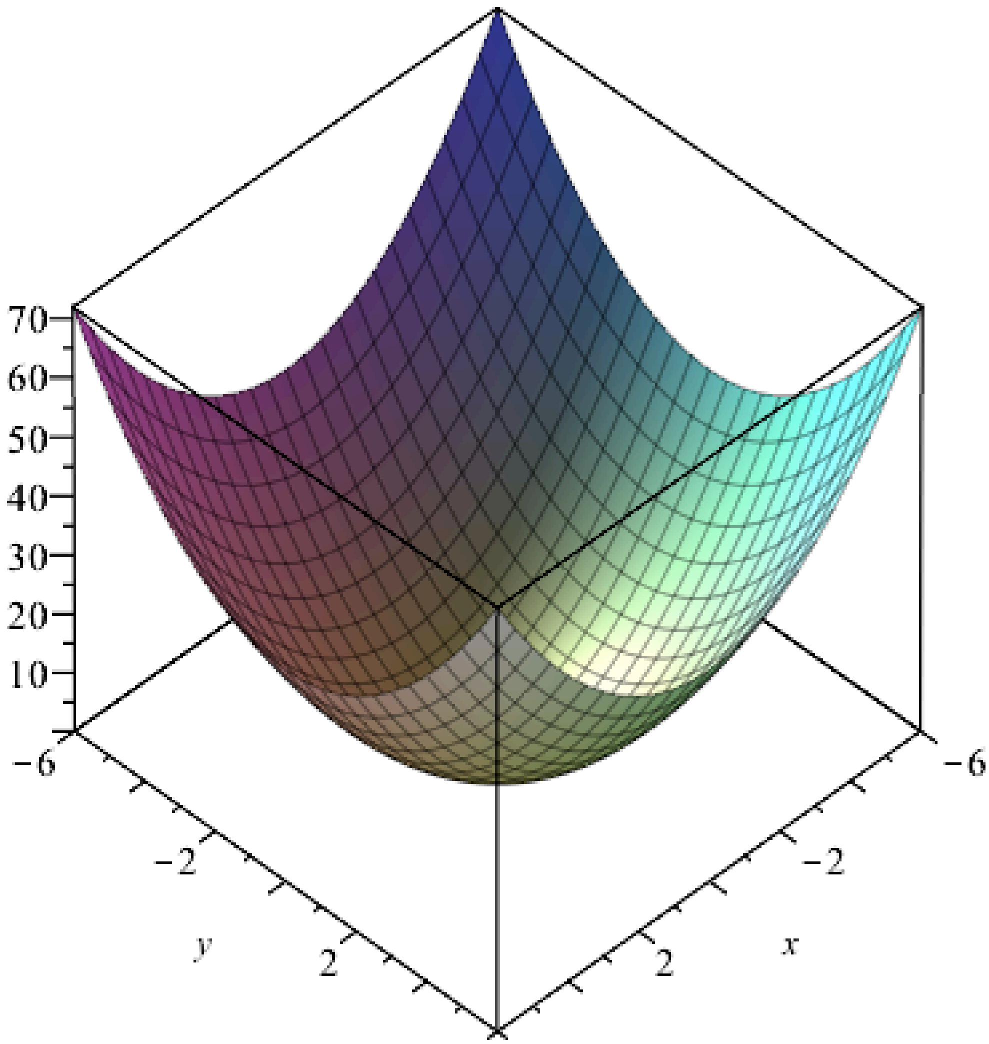
We see that our solutions are circles. We isolate the constant.

```
> C:=solve(soln,_C1);
```

$$C := y(x)^2 + x^2$$

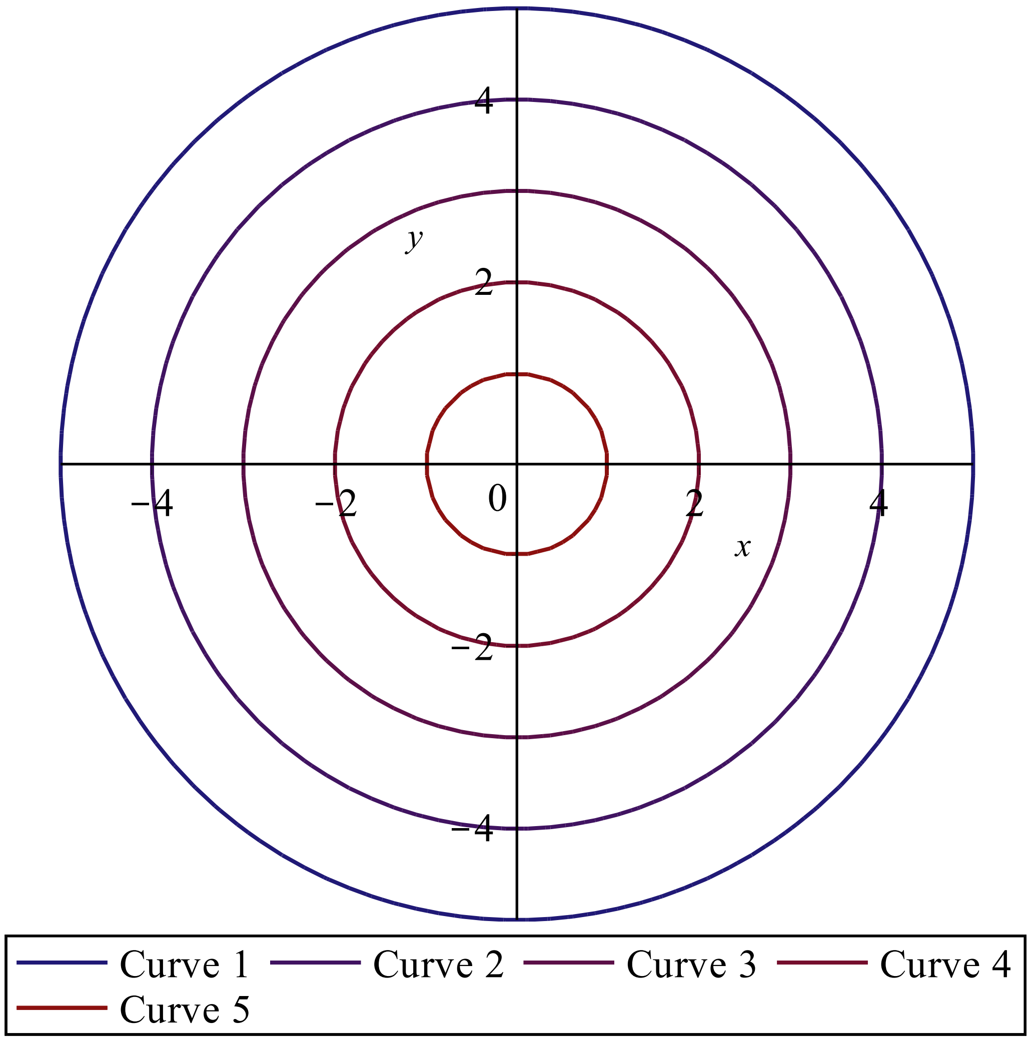
We first look at the three-dimensional plot of $z = F(x, y)$.

```
> plot3d(C,x=-6..6,y=-6..6,axes=BOXED);
```



Indeed, it appears that any level or integral curves would be circles. We use contourplot to draw several integral curves in the *x*-*y* plane corresponding to levels $C = 0, 1, 4, 9, 16, 25$. The option scaling= CONSTRAINED provides each axis with the same scale.

```
> contourplot( C, x=-6..6, y=-6..6, contours=[1, 4, 9, 16, 25],
  scaling=CONSTRAINED);
```



Now let's consider the equation

$$2\,x\,y - \sec(x)^2 + \frac{(x^2 + 2\,y)\,dx}{dy} = 0\,.$$

```
> M(x,y(x)):= 2*x*y(x)-(sec(x))^2;
```

$$M(x, y(x)) := 2\,x\,y(x) - \sec(x)^2$$

```
> N(x,y(x)):= x^2+2*y(x);
```

$$N(x, y(x)) := x^2 + 2\,y(x)$$

```
> deq:=M(x,y(x))+N(x,y(x))*diff(y(x),x)=0;
```

$$deq := 2\,x\,y(x) - \sec(x)^2 + \left(x^2 + 2\,y(x)\right)\left(\frac{d}{dx}\,y(x)\right) = 0$$

```
> odeadvisor(deq,[exact]);
```

$$[\_exact]$$

We see this DE is exact. Let's get the solution.

```
> soln:=dsolve(deq,y(x));
```

$$soln := y(x) = -\frac{1}{2}\,x^2 - \frac{1}{2}\,\sqrt{x^4 + 4\tan(x) - 4\,\_C1},\ y(x) = -\frac{1}{2}\,x^2 + \frac{1}{2}\,\sqrt{x^4 + 4\tan(x) - 4\,\_C1}$$

Now let's get an implicit solution.

```
> soln:=dsolve(deq,y(x),implicit);
```

$$soln := -\frac{\sin(x)}{\cos(x)} + x^2\, y(x) + y(x)^2 + \_C1 = 0$$

We solve for the constant so that we have our solution in the form $F(x, y) = c$.

```
> C:=solve(soln,_C1);
```

$$C := -\frac{x^2\, y(x)\cos(x) + y(x)^2\cos(x) - \sin(x)}{\cos(x)}$$

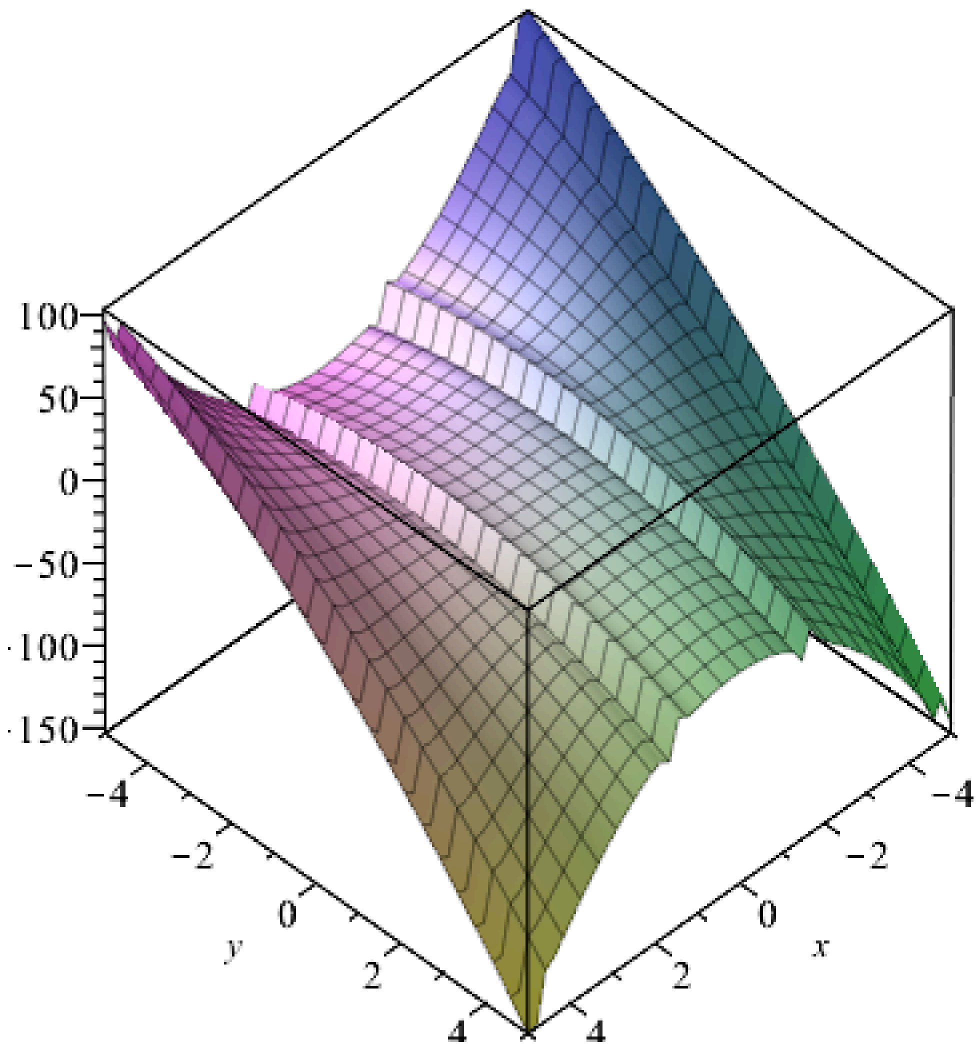
For plotting, we replace $y(x)$ by y.

```
> z:=subs(y(x)=y,C);
```

$$z := -\frac{x^2\, y\cos(x) + y^2\cos(x) - \sin(x)}{\cos(x)}$$

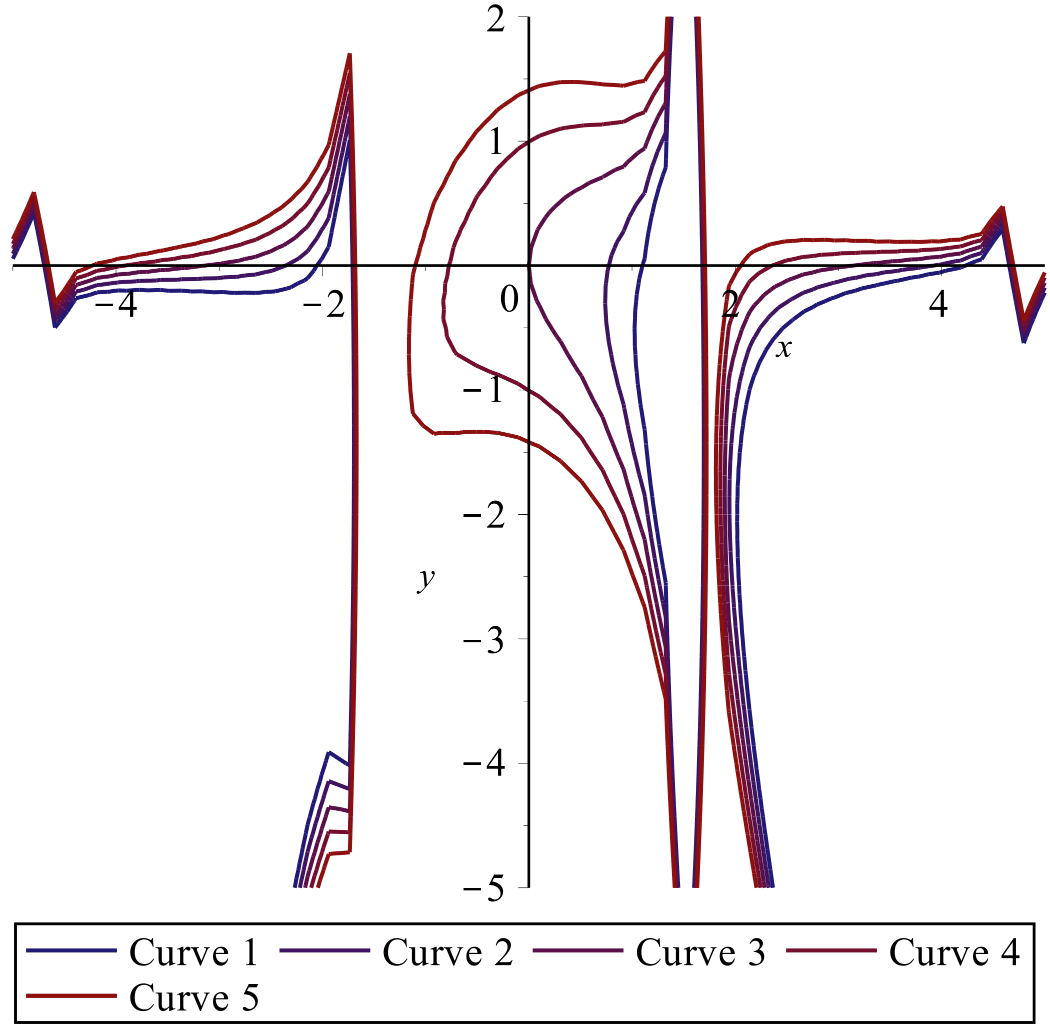
We first look at the three-dimensional plot of $z = F(x, y)$.

```
> plot3d(z,x=-5..5,y=-5..5,axes=BOXED);
```



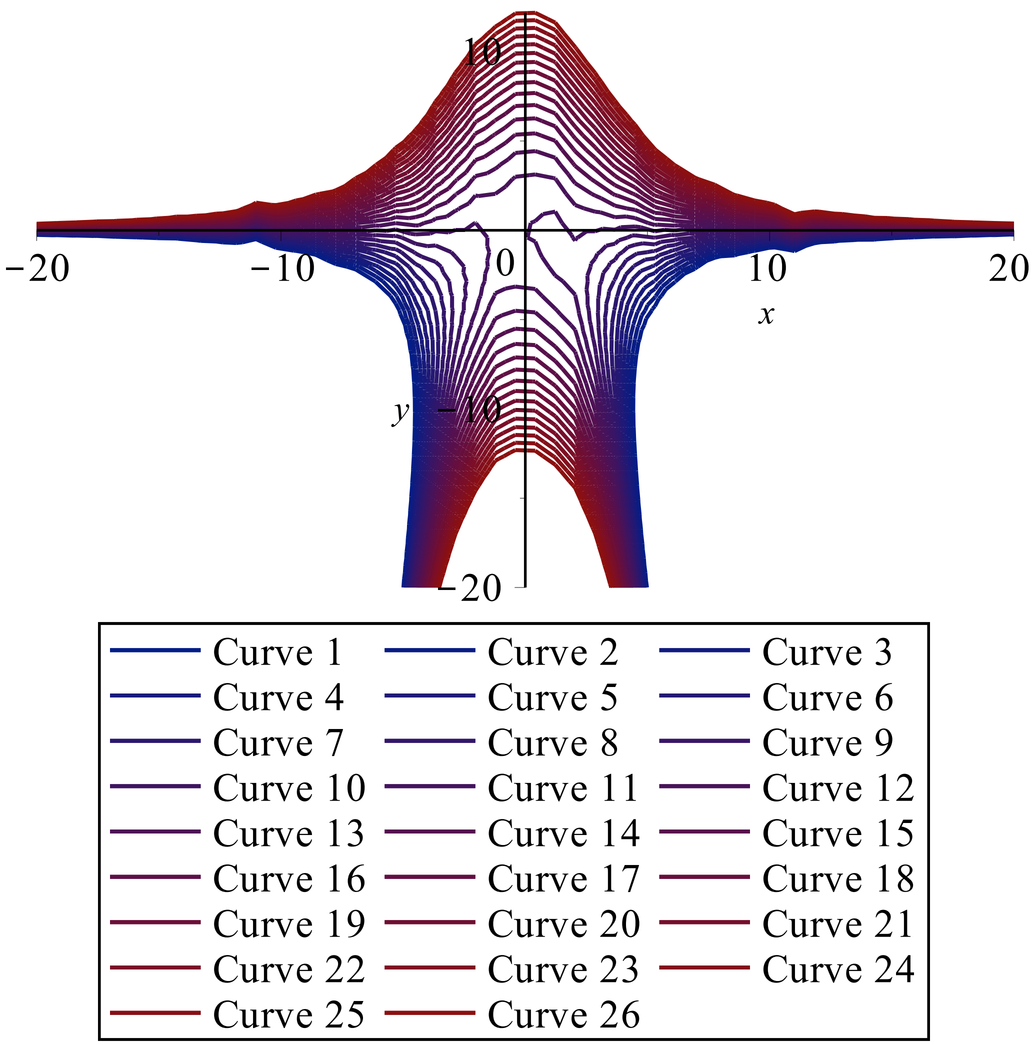
Next we use contourplot to look in close for values of C = -2, -1, 0, 1, 2.

```
> contourplot( C, x=-5..5, y=-5..2,contours=[-2,-1,0,1,2]);
```



Again using contourplot, we expand our view and plot 26 integral curves for the DE, corresponding to values of C = -150, -100, ..., 90, 100.

```
> contourplot( C, x=-20..20, y=-20..20,contours=[seq(10*i,i=-15..10)]
  );
```



Again, recall that the bluer contours reflect higher values of C.