Gaussian Quadrature

We wish to use Gaussian quadrature to approximate the integral \( \int_{1}^{1.6} \frac{2x}{x^2 - 4} \, dx \).

We load the `orthopoly` library to get the Legendre polynomials needed for Gaussian quadrature.

```maple
Digits := 15
with(orthopoly);
G, H, L, P, T, U
```

It is the \( P \) that interests us. Let's look at the first six Legendre polynomials. Note that they differ from those in the text by a constant factor. Since we are only interested in their roots, that is not a concern for our purposes.

```maple
for i from 0 to 5 do
  legedre[i] := P(i, x)
end do;
```

We choose the degree of the Legendre polynomial we wish to use.

```maple
n := 5
```

Get the appropriate Legendre polynomial.

```maple
legendre := P(n, x);
```

We find the roots of the \( n \)’th Legendre polynomial, all of which are between -1 and 1. These are given as a table in the text on page 141.

```maple
r := evalf(allvalues(RootOf(legendre)));
r := [0., 0.538469310105681, 0.906179845938666, -0.538469310105681, -0.906179845938666]
```

We find the corresponding coefficients for Gaussian quadrature as given by the Theorem in the notes. These are also given on page 141.

```maple
for i from 1 to n do
c[i] := 1;
end do;
```
for j from 1 to n do
  if (i<>j) then c[i]:=c[i]*(x-r[j])/(r[i]-r[j]) fi
od:
c[i]:=int(c[i],x=-1..1):
od:
printf(" r[i] c[i]\n");
printf(" ---- ----\n");
for i from 1 to n do
  printf("%18.15f %18.15f\n",r[i],c[i])
od;

We enter the function to integrate as a Maple function to make the change of variables easier.

> f:=x->2*x/(x^2-4);
f := x \rightarrow \frac{2 x}{x^2 - 4}

We give the limits of integration.

> a:=1;b:=1.6;
a := 1
b := 1.6

We do the change of variables so as to integrate from -1 to 1.

> g:=f(((b-a)*t+(b+a))/2)*(b-a)/2;
g := \frac{0.600000000000000 \times (0.300000000000000 t + 1.300000000000000)}{(0.300000000000000 t + 1.300000000000000)^2 - 4}

> g:=unapply(g,t);
g := t \rightarrow \frac{0.600000000000000 \times (0.300000000000000 t + 1.300000000000000)}{(0.300000000000000 t + 1.300000000000000)^2 - 4}

We find the integral by Gaussian quadrature.

> integral:=evalf(sum('c[i]*g(r[i])','i'=1..n));
integral := -0.73396917508020

Compare to the value from the built-in routine.

> Int(f(x),x=1..8/5);
value(%); int \left[ \frac{2 x}{x^2 - 4} \right]_1^{\frac{8}{5}}
\ln(3) + 2 \ln(2) - 2 \ln(5)

> exact:=evalf(%);
exact := -0.73396917508020

Finally, we find the absolute error.

> intererror:=abs(integral-exact);
For Gaussian quadrature, we use the Quadrature command with the option method = gaussian[n] where \( n \) is the number of nodes. The default is \( n = 3 \) if method=gaussian is used. Let us find the same integral as above with \( n = 5 \). For output, we will show value, information, and plot.

```maple
> integral := Quadrature(f(x), x=1..1.6, method=gaussian[5], output=value);

integral := -0.733969175080205
```

```maple
> Quadrature(f(x), x=1..1.6, method=gaussian[5], output=information);

INTEGRAL: Int(2*x/(x^2-4), x=1..1.6) = -0.733969175
APPROXIMATION METHOD: Gaussian Quadrature

--------- INFORMATION TABLE ---------

<table>
<thead>
<tr>
<th>Approximate Value</th>
<th>Absolute Error</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.733969175</td>
<td>4e-15</td>
<td>5.450e-13 %</td>
</tr>
</tbody>
</table>

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Number of Function Evaluations: 50

> Quadrature(f(x), x=1..1.6, method=gaussian[5], output=plot);
```
An Approximation of the Integral of
\[ f(x) = \frac{2x}{x^2 - 4} \]
on the Interval \([1, 1.6]\)
Using Gaussian Quadrature Rule with 5 nodes
Integral Value: \(-0.733969175080201\)
Approximation: \(-0.733969175080205\)